6(2), pp. 74-131, 2008 ISSN 1334-4684

INTERDISCIPLINARY DESCRIPTION OF COMPLEX SYSTEMS

Scientific Journal

Scientific Journal
INTERDISCIPLINARY DESCRIPTION OF COMPLEX SYSTEMS

INDECS, volume 6, issue 2, pages 74-131, year 2008
Published 25. January, 2009, in Zagreb, Croatia
Released online 25. January, 2009

Croatian Interdisciplinary Society (http://www.idd.hr) publishes printed edition (ISSN 1334-4684) and CR-ROM edition (ISSN 1334-5230). Society znanost.org (http://www.znanost.org) publishes online edition (ISSN 1334-4676, http://indecs.znanost.org). Published twice per year. Printed by *Ekološki glasnik, d.o.o.,* Donja Lomnica (HR). This issue is printed in 300 pieces and recorded on 200 CDs.

Journal INDECS is financially supported by Croatian Ministry of Science, Education and Sport.

INDECS publishes original, peer-reviewed, scientific contributions prepared as reviews, regular articles and conference papers, brief and preliminary reports and comments to published articles.

The accessibility of all URLs in the texts was checked one weak before the publishing date.

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EDITORIAL

Welcome to the issue 6(2) of the Interdisciplinary Description of Complex Systems. It contains five texts. Three of them are preliminary reports, while two are regular articles.

All preliminary reports include diverse aspects of complexity and interdisciplinarity, applied onto new areas. The first author, Mr. M. Schatten, brings about new point of view into the problem of formulating in detail the boundaries of living. The second report, written by M. Lisičar et al. applies numerical simulations to work of a specific type of organisations, the testing laboratories. In the report, the authors formulate the basis for future, *in silico* analysis and ranking of possible changes in the laboratories' practices. It is to be emphasised that the authors use the software for numerical simulations which they made, and which is freely available to public. Dr. Türei in the third report raises one important question, about the population in Europe. He set the context in such a way that similar approach could be rather easily adopted to other regions.

In first article, Prof. Zhang investigates the properties of the steady-state equilibrium and the effects of changes in some parameters in a two-sector growth model. Value of the article to be emphasised is its applicability to economic policies in the framework of a two-sector growth model with human capital accumulation. In the second article, Prof. Wolf et al. analyse canonical situation, with a large number of realisations in diverse situations, of a stability of a system with rather generally formulated characteristics.

20 January 2009 Josip Stepanić

LIST OF REFEREES

The following scholars, listed in alphabetic order, refereed manuscripts for the journal INDECS in 2008:

Marta Berčić
Katica Biljaković
Karl Bonutti
Petar Ćurković
Mate Gaal
Željko Grgić
Imre Hronszky
Jerzy Z. Hubert
Timotej Jagrič
Maja Jokić
Boško Kuzmanović
Weisen Li
Marin Lujak
András Margitay Becht
Neli Maria Mengalli
Damir Pajić
Jiancai Pi
Miloš Pukl
Viljem Rupnik
Armano Srbljinović
Janos Szava

their contribution to the quality of the Journal's content is acknowledged.

Zagreb, 20 January 2009

Josip Stepanić

Interdisciplinary Description of Complex Systems 6(2), 74-79, 2008

AUTOPOIETIC THEORY AS A FRAMEWORK FOR BIOMETRICS

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Preliminary report *Received:* 13. October 2008. *Accepted:* 5. January 2009.

ABSTRACT

Autopoietic theory which represents a framework for describing complex non-linear and especially living systems is described in a context of biometric characteristics. It is argued that any living system by performing an internal process of reproducing its structural components yields physical biometric characteristics. Likewise any living system when structurally coupling to another (eventually allopoietic) system yields a behavioral or psychological characteristic of the living system. It is shown that any system that can be considered as autopoietic can potentially be measured, authenticated and/or identified using adequate biometric methods, and thus biometrics is applicable to any autopoietic system: living beings, groups of living beings, social systems, organizations as well as information systems. In the end implications of such a conceptualization are discussed as well as possible applications.

KEY WORDS

autopoiesis, biometrics, structural coupling, component reproduction

CLASSIFICATION

AMS Subject Classification: 68T05, 68T10, 91C99, 92B99, 93B99

JEL: Q57

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INTRODUCTION

Before starting any discussion a brief introduction as well as definitions of basic terms shall be provided. First of all **autopoiesis** is a pseudo Greek word coined from αυτό (auto) for self and πόιησις (poiesis) for creation, production or forming that was first introduced by the Chilean biologists Humberto Maturana and Francisco Varela in 1973 [1] to denote the type of phenomenon they had identified as a characteristic that distinguishes living systems from other types of systems [2]. They claimed that living systems are autonomous entities that reproduce all their properties through their internal processes. Later on this term was introduced into social theory as well as formal organization theory by Niklas Luhmann [3] who claimed that social systems are systems of communication that emerge whenever an autopoietic communication cycle comes into being that is able to filter itself out of a complex environment. Luhman argues that there are three types of social systems: (1) societal, (2) interactional as well as (3) organizational. Any social system has its respective information subsystem described through their communicative processes [4-5]. Systems that are not autopoietic (systems that produce something other than themselves) are considered to be allopoietic (technical) systems.

The term **biometrics** comes from ancient Greek words βιος (bios) for life and μετ ρον (metron) for measure, and thus it represents the measurement of the living. One can approach biometrics in a broader and in a narrower perspective. In the broader perspective biometrics is the statistical research on biological phenomena; it is the use of mathematics and statistics in understanding living beings [6]. In the narrower perspective we can define biometrics as the research of possibilities to recognize persons on behalf of their physical and/or behavioral (psychological) characteristics. We shall approach biometrics in the broader perspective in this paper.

A **biometric characteristic** is a biological phenomenon’s physical or behavioral characteristic that can be used in order to recognize the phenomenon. In the narrower perspective of biometrics physical characteristics are characteristics that one is born with (like a person’s face, iris, retina, finger, vascular structure etc.). Behavioral or psychological characteristics are characteristics that one acquires or learns during her life (like a handwritten signature, a person’s gait, her typing dynamics or voice characteristics). Depending on the number of characteristics used for recognition, biometric systems can be unimodal (when only one biometric characteristic is used) or multimodal (if more than one characteristic is used).

A **biometric structure** is a special feature of some biometric characteristic that can be used for recognition (for example a biometric structure for the human biometric characteristic finger is the structure of papillary lines and minutiae, for the human biometric characteristic gait it is the structure of body movements during a humans walk etc.).

The word method comes from the ancient Greek μεθοδος (methodos) that literally means “way or path of transit” and implies an orderly logical arrangement (usually in steps) to achieve an attended goal [7]. Thus a **biometric method** is a series of steps or activities conducted to process biometric samples of some biometric characteristic usually to find the biometric characteristic’s holder (in the narrower perspective) or a special feature of the biometric sample (in the broader perspective).

A model is a (not necessarily exact) image of some system. Its main purpose is to facilitate the acquiring of information about the original system [8]. A **biometric model** is thus a sample of a biometric system that facilitates the acquiring of information about the system itself as well as information about biometric characteristics. In [9-10] we showed that biometric models consist of biometric methods for preprocessing and feature extraction, quality control as well as recognition.

A sample is a measured quantity or set of quantities of some phenomena in time and/or space. Thus a **biometric sample** represents a measured quantity or set of quantities of a biological phenomenon [10].

A **biometric template** or extracted structure is a quantity or set of quantities acquired by a conscious application of a biometric feature extraction or preprocessing method on a biometric sample. These templates are usually stored in a biometric database and used for reference during the recognition, training or enrolment processes of a biometric system.

Having the basic terms defined one can observe a clear connection between autopoiesis as a framework for describing the living and biometrics as a framework for measuring the living. To underline this connection we shall provide an in depth discussion of component reproduction as well as structural coupling with respect to physical and behavioral biometric characteristics.

COMPONENT REPRODUCTION

Varela gave the following definition of autopoietic systems ([11] adapted from [2]):

> "An autopoietic system is organized (defined as a unity) as a network of processes of production (transformation and destruction) of components that produces the components that:
>
> (i) through their interactions and transformations continuously regenerate and realize the network of processes (relations) that produced them; and
>
> (ii) constitute it (the machine) as a concrete unity in the space in which they [the components] exist by specifying the topological domain of its realization as such a network."

One should observe that there is a distinction between structure and organization (in Maturana's and Varela's sense). While structure is something that is visible (observable) from the outside, organization is unobservable and inside of the system. Structure comprises of a set of components or elements that are exchangeable (that is components change during time) and the mutual interactions between these components. Organization comprises of the relations between processes that produce these components and is stable over time. One can say that the structure resembles the visible image of the internal (non-observable) organization of the living being.

The important concept that shall be outlined here is the internal component reproduction process. One can easily depict this process in living systems which feed themselves with food from their environment that eventually after certain processes becomes an integral part of the living being, facilitating thereby the regeneration of the process.

Thus the only thing that can be observed and likewise sampled or measured from an observer is the structure of components reproduced by the internal organization of the autopoietic system. Since these components are considered to be integral parts of the system they will comprise biometric structures due to complex non-linear processes inside the system. These structures yield physical biometric characteristics. Using adequate biometric methods these structures can be sampled, processed and recognized. Depending on the uniqueness of the internal processes as well as environmental factors these characteristics will be more or less unique for the measured system.

STRUCTURAL COUPLING

The connection between an autopoietic system and its environment is denoted as structural coupling (shown on Fig. 1). "The result of structural coupling is an autonomous and strictly bounded system, that has nevertheless been shaped extensively by its interactions with its environment over time, just as the environment has been shaped by its interactions with the system" [12].

The interactions between an autopoietic system and any other (eventually allopoietic) system shape the behaviour of the autopoietic system during time depending on the characteristics of the environment. If an autopoietic system is structurally coupling to another autopoietic system, language emerges. Language is an important feature between autopoietic systems that is a requirement for the emergence of groups, social systems, organizations and information systems.

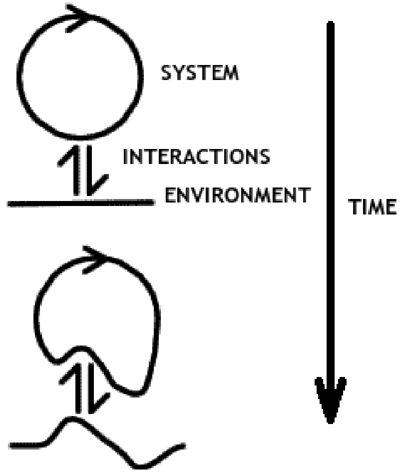

Figure 1. Structural coupling [12] (reproduced with friendly permission of T. Quick).

Thus there will be a clear connection between the autopoietic system's internal characteristics and the characteristics of the environment. This connection is usually described through behavioural or psychological biometric characteristics. Depending on the intensity of structural coupling as well as the on the uniqueness of the internal processes of the system these characteristics will be more or less unique.

This argument lets us conclude that any autopoietic system when structurally coupling for a certain amount of time to another system will acquire some behavioural or psychological characteristics which in turn depend on the environmental system. This is especially true for any human-machine interactions (e.g. typing dynamics, mouse move dynamics, hand grip characteristics).

DISCUSSION

As one can see from the previous discussion there is a clear connection between concepts form autopoietic theory and biometric characteristics. The internal process of component reproduction of an autopoietic system yields physical biometric characteristics whilst the external process of structural coupling between the system and its environment yields behavioural or psychological biometric characteristics.

While this is obvious when talking about living systems and especially humans, this connection implies a whole new field of research in biometrics when taking other autopoietic systems into consideration. The previous discussion showed that one can apply insights from biometrics to groups of living systems (e.g. swarms, flocks), social systems (groups where living systems are mostly humans) with respect to societal (villages, towns, cities, communities, etc.), organizational (companies, syndicates, teams, etc.) and interactive (demonstrations, concerts, happenings, chat rooms, etc.) social systems as well as their respective information systems.

Thus one can measure (sample), process (preprocessing, feature extraction) and recognize any autopoietic system on behalf of their physical and behavioural characteristics.

POSSIBLE APPLICATIONS

As indicated above the connection between autopoietic theory and biometrics implies a whole new area of research for biometrics, but an eventual application area as well. One can observe the main areas of biometrics application in biology and medicine as well as information system's security. By introducing autopoietic theory a new light is thrown on this interdisciplinary field allowing us to apply insights from biometrics in other fields like social and organization theory as well as information systems.

Since, according to Luhmann [3], social systems are autopoietic there are potential application areas of biometric methods in social phenomena. One could use biometric methods (like pattern recognition) to identify social phenomena (e.g. crime, social instability, war) by measuring certain social processes and avoid such unwanted events or facilitate wanted ones.

On the other hand organizations could use biometric methods to identify their certain wanted (teamwork, organizational learning, individual initiative etc.) or unwanted behaviour (groupthink, plant blindness, employee fluctuation etc.) inside them to facilitate or avoid it, respectively.

Of course on should have in mind that this work is theoretical in nature and thus requires additional efforts to find application areas that would be potentially interesting for industry. Such efforts are subject to future research in this area.

CONCLUSIONS

A clear connection between autopoietic theory and biometrics was provided using the concepts of component reproduction and structural coupling. It was shown that the internal process of component reproduction of autopoietic systems yields physical characteristics. Likewise the external process of structural coupling yields behavioural or psychological characteristics.

This new framework allows us to define the difference between physical and behavioural biometric characteristics in the broader perspective of biometrics as follows:

(i) **Physical biometric characteristics** are special features of a biological phenomena's structure which are derived through the internal process of componental reproduction.

(ii) **Behavioral or psychological biometric characteristics** are special features of a biological phenomena's behaviour which emerge due to the external process of structural coupling to environmental systems.

The clear connection implies that any autopoietic system can be subject to biometrics and represents a paradigm shift from traditional systems security perspective as well as other approaches hidden under terms like biometry, biological statistics, biostatistics, behaviometrics etc. to new fields of research and creates a bridge between biometrics and sociometrics as well as other social and organizational sciences.

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AUTOPOIESIS KAO OKVIR ZA BIOMETRIKU

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SAŽETAK

U radu je sa stajališta biometrijskih karakteristika opisana autopoietska teorija, okvir za opisivanje kompleksnih nelinearnih i posebno živućih sustava. Diskutirano je kako bilo koji živući sustav izvođenjem unutarnjih procesa reproduciranja svojih strukturalnih komponenti rezultira fizičkim biometrijskim karakteristikama. Slično tome, bilo koji živući sustav strukturalno povezan s drugim sustavom (alopoietskim) rezultira karakteristikama ponašanja i psihološkim karakteristikama živućih sustava. Pokazano je kako bilo koji sustav, kojeg se može smatrati autopoietskim, može biti mjeren i identificiran primjenom prikladnih biometrijskih metoda. Zbog toga je biometrika primjenjiva bilo za koji autopoietski sustav: živa bića, grupe živih bića, organizacije kao i informacijske sustave. Na kraju su diskutirane posljedice takve konceptualizacije kao i moguće primjene.

KLJUČNE RIJEČI

autopoiesis, biometrika, strukturalno vezanje, reproduciranje komponente

Interdisciplinary Description of Complex Systems 6(2), 80-87, 2008

SYSTEM OF NON-DESTRUCTIVE TESTING

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Preliminary report *Received:* 19. November 2008. *Accepted: 19*. January 2009.

ABSTRACT

In this paper the initial analysis of NDT laboratory is presented by means of agent-based modelling. For the purpose of analysis, laboratory is taken into account as a complex system consisting of three agents; equipment, personnel and specimens. Interaction between the agents is circular. In that sense, the agents are mutually interconnected in a way that one agent simultaneously interacts with others.

According to the interactions specific for NDT laboratories, the response of total testing time is presented considering various number of laboratory personnel while each operator has different skills and ability. Agent personnel has to perform testing of specimens. Since the complexity of specimens is quite diverse the specimens are represented as an agent. Additionally, during the whole time sequence of testing a certain specimen, operator is using NDT equipment relevant for the testing method, while the particular time of usage of the equipment can be shorter than the whole testing time. Availability of the equipment is therefore another agent. The evaluated outcome is the total testing time.

Presented results are obtained carrying out a simulation by means of multi-agent modelling and simulation tool named "ENTORAMA". Finally, the overall laboratory's performance is given in the respect of the number and structure of the laboratory personnel.

KEY WORDS

non-destructive testing, personnel, equipment, laboratory, agent-based model

CLASSIFICATION

JEL: C65, L15

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INTRODUCTION

The structuring of activities in testing laboratories is important contribution to the overall quality management. There is permanent need to test whether the adopted structuring is optimal in a given working environment. However, methods used in that task, prevalently experiments or mental models, are not sufficient to provide the clear solution to the problem.

Nowadays quality assurance requires to determining which technique should be used, and when and where it will be applied. It is necessary to implement an appropriate technique if one wants to ensure an adequate result with the adequate level of assurance. Thus, non-destructive testing (NDT) has taken the responsibility, as a part of preventive technologies, for detecting discontinuities and predictions of further improvements in industrial production. It is also an unavoidable mean to verify and detect possible discontinuity within in-service inspection of plants, materials and structures, thus preventing accidents that could have major negative impact on environment [1, 2].

Moreover, the non-destructive testing techniques are used to assess the integrity of the components or structures throughout their designed life-time. To summarise, the main contribution of non-destructive testing is to provide confidence in obtained results to customers [3, 4].

For initial discussion, a model of laboratory is emphasised as a crucial next step in improving the structuring of activities in testing laboratories. The model represents a NDT laboratory in an idealistic case where the personnel are reduced. Only characteristic of personnel that is represented is their ability and competence, no other attributes are included. Further upgrade is provided for implementation all disability of indicated elements.

In this article, the agent based simulation model is performed in order to research behaviour of complex organisation that is representing as laboratory for non-destructive testing. In accordance with formulated relations between agents it is demonstrated how the outcomes depend on actions between entities and how the system efficiency is changed with personnel's characteristics. Because of the flexibility of the agent based modelling, it is useful to find out which of part has significant contribution in systems ability and which of part contribute the lowest influence. Following that, the system can be improved and the expected quality assurance level can be reached.

MODEL

A laboratory is a representation of a complex system. Therefore, a reliable model of a laboratory will include elements of a complex system description. In this model, we extract several elements of a typical NDT laboratory, and relate them mutually with specific relations, bringing about summarily representation of laboratory as a complex system.

We model the dynamics of these elements in a given working environment using an agent based model approach [5, 6]. In our approach, the laboratory consists of the following elements: equipment, personnel, specimens and procedure.

Equipment is essential in organising all other activities in chain of complex system. It does not mean that expensive contemporary equipment will produce better result. The material facilities determine all other activities in procedure to detect presumably discontinuity. To provide systems' quality, the equipment should be calibrated and certified and it is necessary to do proof and evidence of their accuracy and uncertainty. Periodic calibration, that is required and determined by codes and therefore implemented in system, is essential condition in estimation measurement uncertainty that is prescribed to device. It is also unavoidable part of equipment to have a reference block to ensure traceability and to provide systems' adequacy [4, 7].

In our approach equipment is an agent consisting of all items like instruments, probes, reference blocks and material means for enabling the personnel to perform operations as are prescribed in written testing procedures. In this approach there are five different NDT methods with different equipments. Thus, the equipment is distinct for visual testing (VT), liquid penetrant testing (PT), magnetic particle testing (MT), ultrasonic testing (UT) and radiography testing (RT). The variables of the equipment are current number of personnel that share the same equipment and maximum number of personnel that can share the same equipment. How the equipment will be shared depends on method and handling time. Thereby some of equipment can be used by several operators in one time scale. One of them is penetrant testing (PT) with testing time around 30 minutes whereof the equipment is used only 5 minutes. Visual and ultrasonic methods have just testing time whereas the equipment is used all the time with duration of 30 minutes for visual testing and 60 minutes for ultrasonic testing. Testing time to complete the magnetic method is around 10 minutes what include 5 minutes for equipment using and finally the on-site radiography method need 90 minutes for testing where only 10 minutes the equipment is used. The method that is applied on one specimen depend on its complexity, but regardless on complexity, minimal number of testing method for one specimen is two. In that sense and in accordance with most technical codes of practice, the method that is always applied is visual testing and is specified with 100 % applicability. Estimated testing probability for application of volume method and surface method is 70 %. The volume method refers on ultrasonic testing and radiography testing where each applicability is 35 %. Surface method includes magnetic testing and penetrant testing with same applicability of 35 %.

Next essential condition for functioning of every type of a system is personnel. There is major need for continually improvement personnel' skills within the meaning of continuous training [2-4]. The personnel should be well motivated because their satisfaction can have positive influence on system. The NDT method that can be applied depends on components' characteristics as well as on the physical characteristics and properties of the material. It means that staff should know how to perform testing correctly without skipping any part of procedure. If the method is adequately performed then possibly existing discontinuity will be detected.

In this paper personnel is agent which represent people performing regular NDT operations of a laboratory, i.e. examinations and measurements, receiving the samples and conduct prescribed testing, writing records and reports, ensuring that results are accurate, reliable and timely. They provide crucial information for detecting and evaluating discontinuity and defect concerning acceptance criteria given by relevant codes or standards.

The variables of the personnel are characteristic of experience, understanding codes and standards and number of hours that is spent in testing.. According to their operating ability and testing experience, the personnel are divided into two categories, A and B. Personnel in A category are less competitive with lower level of skills. The opposite is B category including the personnel with better abilities and competences. In further discussion the personnel categories will be attributed values 1,5 for category A and 1 for category B.

Specimens that are under testing should be carrying on properly. The laboratory has the responsibility to provide adequate storage for all specimens that comes into laboratory. It is important to send back the specimens in unmodified state as this testing is non-destructive. If the customer does not have any particular request for testing method, the personnel will decide and choose which of the test method will be applied and performed according to relevant technical specification.

Specimen is an agent that is sent to the laboratory which has to detect possible discontinuity. The variables of the specimens are specimen complexity and discontinuity presence. In general specimens have discontinuity but it does not mean that it is defect. The specimens' complexity is quantified as continual variable between 1 and 2. The most complex specimens will be quantified with number 2 opposite to 1 which regard on the simplest one. Some testing will be carried out outside the laboratory since the most complex specimens require on-site inspection. In that testing condition the operator with better ability will perform testing faster and more accurate achieving the same level of testing reliability as operator with lower ability. Some of specimens will not have any discontinuity and so neither defect. But this issue is not emphasizing like separately variable in this concept. Total number of specimens in laboratory is divided in appropriate ratio between category A and B.

The written NDT procedure should ensure high level of detection probability what is in correlation with reliability of results. Specific steps for every method are defined and finding of any significant discontinuity is ensured. For laboratory quality system it is important how the records and documentation are carried on and are they in accordance with codes and references. In our model of simulation procedure is not taken as a particular agent but it is implemented in agent personnel concerning personnel's competence and technical knowledge with reference to performing testing.

Environment is element that will not have noteworthy influence on test results if the conditions are under control. Effect of environment is excluded from observed laboratory here representing complex system.

INTERACTIONS

In our laboratory model, interaction between the agents is circular. In that sense, the agents are mutually interconnected in a way that one agent is simultaneously in interaction with other two. In other words, it is by no means straightforward to monitor interaction between just a two of agents' types (Figure 1.).

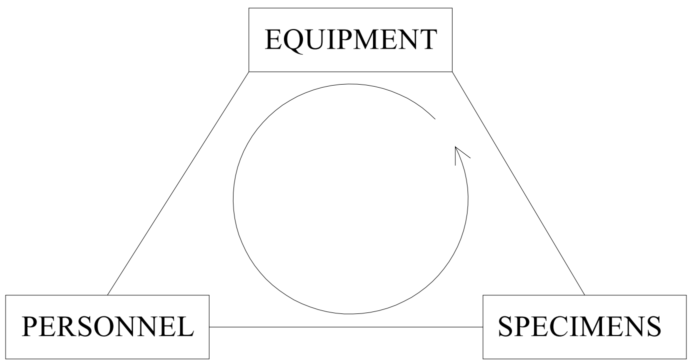

Figure 1. Circular interaction.

In the model, agents interact in the following way: personnel interact with equipment in a way that time of using equipment depends on quickness of personnel and method that is applied. Personnel can start with testing only when the equipment is available. Some of the equipment can be used by several persons what depends of method that is performed. Therefore for each method there exist both the associated time for using equipment and the total time for testing. In that way, total time for testing whole specimen is maximally reduced whereas the equipment after using by one person is forwarded to another. It is obvious that testing time also depends on personnel skills, competences, knowledge and their adequacy. But this testing time is more noted in interaction between specimens and personnel. Regarding the specimens' complexity it is clear that an operator with more testing experience and proficiency will perform testing faster opposite to operator with less ability. As it was emphasised before, this system is circular, thus an indirect interaction of equipment and specimen is left. Their interaction is evident in time that is longer if the specimen is more complex. Therefore, for their testing it is necessary to perform several methods what extends total testing time. Some of the specimens need to be tested by more, e.g. 4 to 5 different methods. None of the specimen requests only one testing method, i.e. the minimum number of methods applied to a specimen is two.

SIMULATION OF COMPLEX SYSTEM

As a point of reference, at first we develop the ideal case in which there are no restrictions on equipment availability. In that case all personnel would have their own equipment for testing. They will perform testing on a specimen chosen randomly. Duration of testing is calculated using following expressions:

$$T_i = \sum t_M \, p_M \cdot p_i \cdot s_i , \tag{1}$$

$$s_i = \frac{\sum_j s_{i,j}}{\sum_j 1}, \; s_{i,j} \in \{1;2\}, \tag{2}$$

$$T_0 = \sum_i T_i , \tag{3}$$

in which t_M is the average time for testing specimen with given method, p_M the probability for utilising a given method, the index I denotes a personnel's category while {j} denotes a set of specimens with different complexity, T_i is the average testing time performed by one person on average specimen s' complexity, P_i personnel adequacy (skills, competences, capability, knowledge), s_i is pecimens' complexity and T_0 the total testing time.

Expressions (1-3) form set for ideal case where testing time depends on specimens' complexity and personnel adequacy. Average time for testing one specimen is attributed for every method. It is determined as an average time needed for testing that is performed on one specimen with ordinary complexity by personnel of sufficient competence and skills.

Beside presented ideal case the simulation for a real case laboratory has been performed. For a single laboratory with different number of personnel that are allocated into category A and B we performed simulation in the following way: for each ratio of personnel in categories A and B, the simulation is repeated 100 times, and as a result we took the average value of these simulations. The result of the simulation is duration of testing time where specimens' complexity, personnel adequacy and testing method are varied. Duration of that time is given by the following equation:

$$T_\Sigma = \sum_i T_i + \Delta , \tag{4}$$

in which T_Σ is the testing time obtained by simulation and Δ denotes duration of higher order effects which measures the influence of combinations of elements.

The results that are obtained from expression (4) refer to a real case in which the personnel and their operating time depend on equipment ability.

RESULTS

Outcomes of the model are time duration of testing performed by personnel with different capability and competence. Time duration of testing is consequence of specimen's complexity and of personnel capability and competence. Personnel that perform testing randomly take the specimen. Personnel with less skills and competence will perform testing longer, with enlarged possibility for mistake. Their testing time and time of equipments using will be longer and their behaviour will have larger influence on system. So, in this approach the personnel have the biggest influence on system especially on testing time and system efficiency.

Restrictive element in laboratory system is number of equipment and its operative time. The time needed to perform each method is quantified with minutes. Some of the equipment can be consecutively used by several persons during the simultaneous testing cycle what depend on method that is performed.

Total testing time that is given by expression (3) is set for the ideal case which is independent as regards personnel number in laboratory. For each ratio of categories A and B (personnel adequacy), testing time needed to perform testing will be the same regardless to total number of personnel in laboratory. This is expected effect as there is no factor affecting personnel interaction since each operator possesses own equipment. Consequently the same results will be obtained for any combination of personnel in categories A and B. In Figure 2 the dependence of total testing time on percent of personnel from category “A” (thus also with personnel from category “B”) is shown. If the laboratory has 40 % personnel from category “A”, than 60% personnel will be from category “B”.

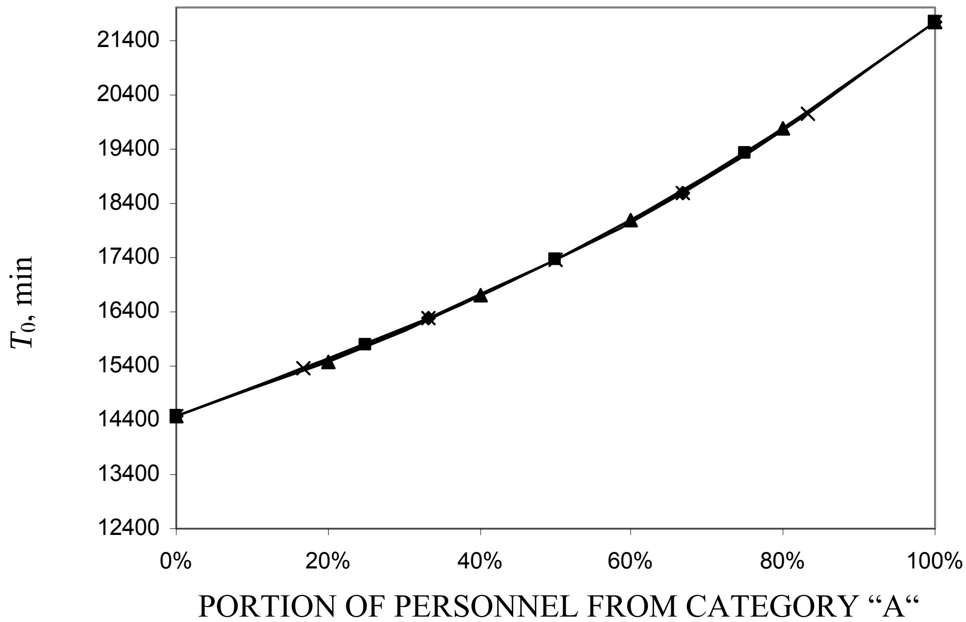

Figure 2. Testing time in ideal case.

In the model, four different cases are simulated where "case" implies laboratory with certain number of personnel allocated in categories A and B. The simulation is run for four laboratories equipped with personnel consisting of 3, 4, 5 and 6 operators in total.

The obtained simulation results reveal that laboratory with more operators will consume greater total testing time rather than laboratories that have less operators while. Respectfully corresponding curves for different laboratories are shown in Figure 3. In that sense total testing time does not mean longer period of testing for the reason that total testing time could be distributed to more personnel that will finally result with shorter period of testing.

As is shown in Figure 3, for a particular laboratory the total testing time appears longer in comparison with testing time in ideal case. It is due to the fact that there are more operators sharing the available equipment.

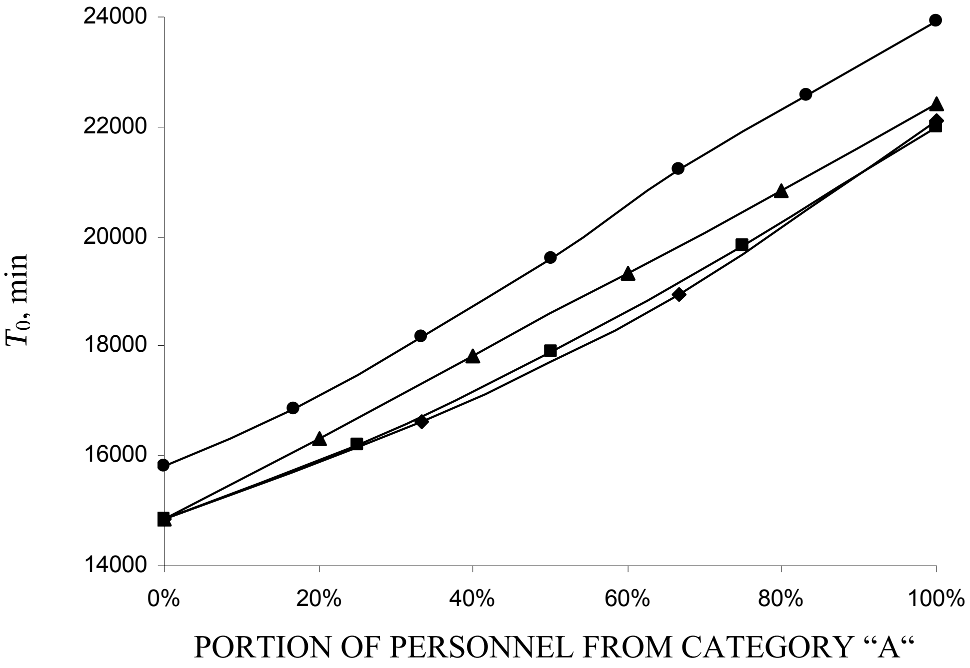

Figure 3. Testing time obtained by simulation.

CONCLUSIONS

In this article we presented initial results regarding the influence of the number and structure of the laboratory personnel on the overall laboratory's performance. We differentiated between the ideal and realistic case. In the former, the total time needed to process a given specimen set is solely sum of times which individual operators need for testing of randomly attributed specimen. In the later case, total time is duration of processing as given in the simulation using the agent based model of the laboratory. The difference between two respective times is caused by impossibility for sharing the equipment. In ideal case personnel have their own equipment without need for sharing them what is not situation in realistic case. If the laboratory has larger number of personnel, difference will be larger. Larger number of personnel is resulting in increasing waiting time for equipment availability what is actually inefficient time generating additional costs for laboratory. In the simulation we introduced time needed to perform testing. For a given set of specimens the distribution of required testing methods was prescribed in accordance with the common practice.

By means of presented simulation and developed model of laboratory structure and processes the laboratory management has a convenient tool for analysis, review, planning and improvement of laboratory resources. Simulation and model also provide indicators as a measure for improvements of quality system and economic benefits.

Simulation results revealed that further research must be focused more in detail on the interaction between available laboratory testing equipment and corresponding number of personnel with different testing abilities.

ACKNOWLEDGMENTS

Work presented in this article and results obtained are part of research project "Reliability of non-destructive testing methods", No. 120-1201767-1763, funded by Croatian Ministry of Science, Education and Sports.

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SUSTAV NERAZORNIH ISPITIVANJA

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SAŽETAK

U radu se laboratorij analizira kao kompleksni sustav s tri vrste agenata: opreme, osoblja i uzoraka. Međudjelovanje između agenata je cirkularno. Na taj način, agenti su međusobno povezani tako da svaka vrsta istovremeno međudjeluje s ostale dvije vrste. Prema njihovom međudjelovanju, određeno je kako vrijeme ispitivanja uzoraka ovisi o ukupnom broju osoblja različitih vještina i sposobnosti. Agent osoblje provodi ispitivanja na uzorcima različite složenosti. Značajno pitanje u svakom laboratoriju je oprema. Ako osoblje mora dijeliti istu opremu, ukupno će trajanje ispitivanja biti veće nego u laboratoriju u kojemu svatko od osoblja posjeduje potrebnu opremu. U tom slučaju će trajanje ispitivanje biti najkraće, što je u radu označeno kao "idealni slučaj". Idealni slučaj se uspoređuje s realističnim slučajevima koji su razrađeni koristeći modeliranje pomoću agenata i programski paket *Entorama*. Na taj način izdvojen je utjecaj broja i strukture laboratorijskog osoblja na ukupno djelovanje laboratorija.

KLJUČNE RIJEČI

Nerazorna ispitivanja, osoblje, oprema, laboratorij, modeliranje pomoću agenata

Interdisciplinary Description of Complex Systems 6(2), 88-94, 2008

CAN WE SLOW DOWN THE ALARMING RATE OF EUROPE'S AGEING?

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Preliminary report *Received:* 6. April 2007. *Accepted:* 16. December 2008.

ABSTRACT

In this report I call attention to the ageing of Europe's community and the disadvantageous effect of ageing. I list methods that could be applied to moderate this process and propose some alternative solution laying much more stress upon the replacement of population.

KEY WORDS

Ageing, population, Europe

CLASSIFICATION

JEL: J11

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INTRODUCTION

The demographic processes are among the most important factors influencing a particular region's growth opportunities. Demographic processes are dependent on typically slowly changing but basic factors such as the values and habits of strata and individuals, or the institutional framework and interests in a society. The impacts of demographic processes become visible only in the very long run, often reflecting decade- or even century-long tendencies hence their huge inertia. Consequently, they can be predicted with a high level of certainty, and if adverse changes are prognosticated, only a concerted and strong series of acts can yield favourable results.

When discussing the subject, we may not ignore the fact that countries coping with population fall are characterised by the exaggerated and prodigal consumption, the consumerism, thus a disproportionately high value of "ecological footprint"[1], compared to the global average. Consequently, a population rise in these regions would work towards unsustainability, or in other words, towards the more or less well-grounded vision of a global disaster [1] more intensively than it would in the developing countries. From this aspect we may regard population decrease as a favourable process in these areas, for this is one of the preconditions for the society to avoid bumping into its own ecological limits. However, in the following decades a no less exciting issue will be how to handle problems triggered by population decrease, and this is what I wish to deal with in my writing.

Even today, the explosive increase in population defines the limitations of sustainability and balance in many "developing" countries. Nevertheless, despite the fact that this is likely to remain the core problem in the future as well, by today the situation has changed in many aspects. As is shown in Figure 1, there was a turning point in the growth of the world's population in the 90s. Up to that time, the world's population grew at an accelerating pace, while in the 90s the tendency took the opposite direction, i.e. the population growth was decelerating. This change took place earlier in the developed countries, so that in Europe and Japan the aging of the population has posed an ever-increasing challenge for society for quite some time.

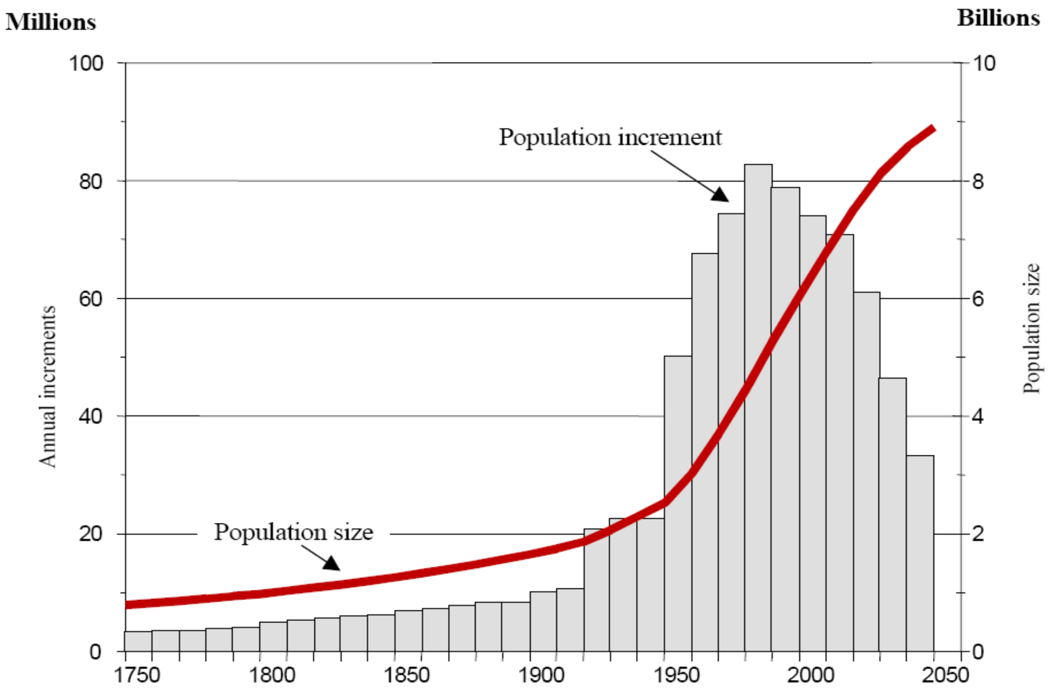

Figure 1. Long-term world population growth, 1750 to 2050 (source UNDP).

PROBLEM DESCRIPTION

The problem is not only that people live longer in the developed world, which changes the age composition of the population. The major cause of aging is the extremely low fertility rate, which is about 1,4 in the European countries, well below the replacement level, 2,1. In most European countries the number of new-borns remains well below that observed in earlier generations, so that without immigrants the population of European countries would go down by nearly 1 million each year.

What is more, according to forecasts, Europe has not touched the bottom yet [2]. By 2050 several countries in Europe will face a considerable population fall. For example, the decrease in Ukraine's population will reach 20 million (40 %), in Belarus or Bulgaria 30 %, while in some Eastern and Middle European countries a 15-25 % population drop is projected. The problem is not confined to the former socialist countries. Europe at large is also facing the problem of decreasing population, even though to a lesser extent. In 2050, the number of Italians and Spaniards will be 16 million and 9 million less, respectively, than it was at the time of the millennium.

The problem is no that some regions or countries will be less populous tomorrow: in areas with high environmental load population decrease may even be interpreted as a favourable process. Notwithstanding their decreasing population, most of the countries in Europe will still be among the most densely populated areas of the world. The problem is rather represented by the decline in population itself, and all its consequences. Due to a lower fertility rate and increased life expectancy, the median age of the population rises at an outstandingly fast pace, growing by 10 years in the first half of the 21st century, and probably exceeding 47 years by 2050, that is, at that time over the half of the population will be older than[2] 47. Even more frightening is the outlook that the proportion of those over 85 will be nearly tripled to a value over 11 %, while the proportion of children under 14 will fall by nearly one-fifth, hardly exceeding the percentage of over 85.

According to predictions (EPC and EC (2005)), even counting with a slight increase in the proportion of working population by 2050, the number of working population will fall by some 9 million, mainly due to a 16 % (48 million) drop in the working age population. Meanwhile, the number of over-65s is projected to increase by 58 million (77 %), doubling the old-age dependency ratio to 54,5 % from the present 24,5 %. The so-called effective old-age dependency ratio (meaning the ratio of over-65, non-active and working people) will rise to a similar extent, from 37 % to 70 % between 2003 and 2050. At present, the ratio between working age people and over-65s is 4:1. This proportion will change to less than 2:1.

Figure 2 demonstrates the above discussed changes in age composition. In my view, these changes affect economy in two ways. On the one hand, the elderly population is generally less dynamic, innovative and adaptive, which will have a slowing-down effect on the increase in productivity. Most probably, this factor already has a significant influence on European economy, as the European population is already the oldest in the world, and the pace of increase in productivity has been decreasing for decades in the region [3].

On the other hand, the above changes also result in a rise in the expenses and the emergence of financing issues. The problem is already present, as we find it increasingly hard to finance the rising costs of old-age health care and other related services. According to EU estimations [4], the related extra expenses reach 3-4 % of the GDP. As in Europe the health care and pension systems are almost exclusively financed by the state, the new situation may force governments to increase related spending by at least 10 %, providing the present structure of financing remains.

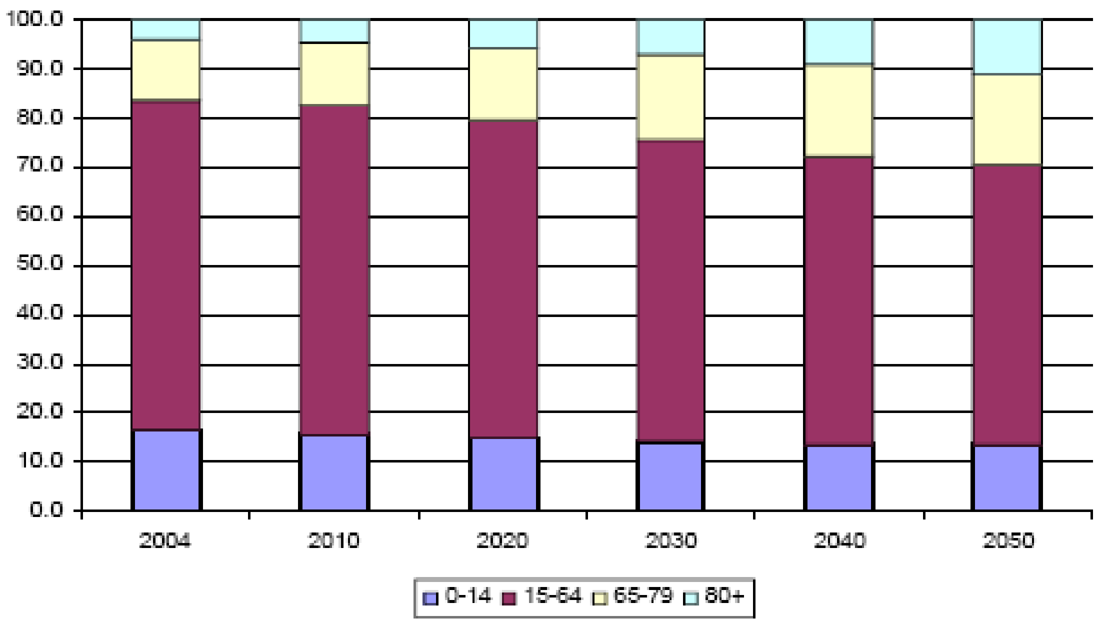

Figure 2. Changes in the age structure of the EU25 population [4].

However, I think that these predictions roughly underestimate the expected extra costs, because, as shown above, the number of over-65s will rise by nearly 80 %, and even if we look only at the number of pensioners, and take a higher age of retirement into consideration, the number of pensioners will still be at least one and a half times higher. Within this group the proportion of over-80s will show a more pronounced increase. This fact suggests a sharp rise in health-care and pension-related expenses. As far as the other end of the scale is concerned, a lower number of younger people, consequently lower educational and dependency-related expenses, but longer educational periods and thus higher relative costs as well should be taken into consideration. However, in general, a major increase in the expenses spent on the inactive population should be expected.

It seems even more true if, stepping beyond a general consideration of the likely increase in expenses, we examine the cost rises in comparison with the dropping number of working population, who practically finance inactive people. From this aspect, the burden on the working population will gradually increase in the form of growing social security expenses and taxes. These are strongly discouraging factors in terms of producing income.

The set of problems outlined above basically stems from the fact that Europe's population does not reproduce itself, European women do not give birth to enough children to maintain the population. Meanwhile, an ever-longer life expectancy is added to the situation. Is there a way to ease the above problems?

DISCUSSION

I see three major ways of solving the problem: First, we may increase immigration [5, 6] from areas outside Europe. At present, the net volume of the increase is 0,3 % per year, which is almost enough to counter-balance the natural decline of the population, but is far from being enough to stop unfavourable changes in age composition and the ageing process[3]. Nevertheless, migration plays an important role in the reproduction of Europe's population. Unfortunately, European societies do not seem to have realised the significance of migration in an area where the working age population is gradually shrinking and the elderly population is massively growing. This realisation must be the first step toward a more open society and labour market, and the integration of migrants and acceptance of cultural diversity. However, fertility is highly culture-dependent, so there may be a real concern that migrants, providing that they adopt local cultural attitudes, will soon prefer having only one or two children.

On the other hand, we can encourage people to work longer and retire at a later age, and this is exactly what European governments and OECD [7] appear to aim at. Its significance is highlighted by the fact that life expectancy of both women and men is predicted to grow by an average of 5-6 years in the EU in the first half of the 21st century [4]. If longer life expectancy will not be accompanied by an increasing number of years at work, the age of retirement will not get higher and the elderly dependency ratio will worsen. As a consequence, the financiability and sustainability of pension systems will become questionable. By contrast, if longer life expectancy does not result in longer time spent in the pension system, i.e. higher overall amount of pension paid, but rather increases the length of the period throughout which people pay contributions, the burden on the population will altogether ease. In fact, it is quite logical that longer life should mean a longer period spent at work, which is to improve the ratio of the active and inactive periods of life.

From a different point of view, the number of active years is limited by the labour market situation, which is likely to be even more influential in this sense than demographical processes [8]. If the amount of pension would be more directly dependent on the actual value of the contribution paid by people and people's expected life length, it is likely that instead of seeking to retire as early as possible, people would rather stay at work for as long as they can. Governments could support this option by a progressive decrease in the amount of contributions to be paid in the last quarter of the expected life length. Of course, business actors might also facilitate the process by concluding contracts in which the amount of pension is established to correlate with contribution payments over a whole life, not just with the income received in the last few years, and also by accepting to a larger extent the possibility of declining wages in the last years of life[4].

Efforts to promote retirement at a later age as well as other ways to increase employment in Europe seems even more arguable if we consider that the main reason for Europe's lagging behind the US is the lower number of years spent at work, though in recent decades the US significantly outstripped Europe both in terms of productivity and its growth rate. This solution could, in spite of the unfavourable changes in age composition, counter-balance the worsening dependency ratio to a certain extent and increase growth pace (or at least decrease the rate of slowing down). Furthermore, a higher level of employment is likely to decrease dramatically income discrepancies in the society, and the level of income redistribution required for the decrease of this discrepancy.

The above two directions are very important in terms of taming the aging process and improving the dependency ratio. However, on the long run, a significant improvement of the fertility rate should also be targeted, that is, we should urge people to raise more children. Long-term results may only be provided by a better fertility rate, even if the rising number of new-borns will increase youth dependency first, and working age population only decades later.

I think that the fertility rate is hard to influence on the long run. This is why more radical and consistent social and welfare policies should be introduced by European governments, instead of the existing methods. At present, governments in Europe try to increase fertility rate by providing benefits and allowances. However, this method is hardly successful (because the fertility rates are so low and their mitigating effects on the income inequalities are insufficient [9]), and drives up state expenses and decelerates economic growth. Despite the redistribution system, the major factor resulting in per capita income disparities between families is the number of children. It is quite logical, as the income provided to children is far bellow the income level of working parents, consequently, raising children means a significantly lower livings standard. The existing income structure is especially responsible for decreasing fertility rates in the countries of Europe. For the causes of the low fertility rates in Europe see [9 – 12].

Financing the benefits provided to families with children from extra taxes paid by those without children, instead of the general pot of taxes and social contributions, would encourage people to raise children in two ways: first, it would put an extra burden on those not contributing to social reproduction, thus making the idea of living without children less desirable; and second, it would allow governments to provide higher allowances to families with children. Not only would this solution encourage people more effectively to have children, but it would also help balance the relative income levels[5].

A society can be considered healthy only if it is able to reproduce itself. If the individuals in a society fail to perform in line with this, it is the government's task to establish criteria that make living without children less desirable, and encourage people to contribute to the social reproduction.

REMARKS

[1]The concept was introduced in the ecological literature by William Rees [13].

[2]The data relate to the European Union, not the whole of Europe.

[3]One of the United Nation's forecasts dealing with demographical and migration trends [14] provides a calculation as to how many immigrants should move to Europe in the first half of the century to maintain population (96 million people), or to keep the ratio between the number of those between 15 and 64 and over 65s (more than 1350 million people).

[4]The role of the resilience of the labour market and wages is highlighted by, for example, studies written by the world bank's authors [15, 16].

[5]An alternative or supplementary solution of the problem may be to let parents have a share in the human capital of their natives.

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MOŽEMO LI USPORITI ALARMANTNU STOPU STARENJA EUROPE?

S. Türei

Odsjek za ekonomiju – Sveučilište za tehnologiju i ekonomiju u Budimpešti
Budimpešta, Madžarska

SAŽETAK

U ovom radu ističem starenje Europskog društva i nedostatke koje donosi opće starenje. Navodim metode koje se može primijeniti radi usporavanja tog procesa i predlažem alternativno rješenje koja postavlja znatno veće zahtjeve na zamjenu stanovništva.

KLJUČNE RIJEČI

Starenje, stanovništvo, Europa

Interdisciplinary Description of Complex Systems 6(2), 95-116, 2008

A TWO-SECTOR GROWTH MODEL WITH ENDOGENOUS HUMAN CAPITAL AND AMENITIES

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Regular article *Received:* 3. February 2008. *Accepted:* 21. January 2009.

ABSTRACT

This paper examines issues related to urbanization with labour migration. The main departures from the traditional approaches to dynamics of economic structures are that the paper uses an alternative approach to consumer behaviour and introduces human capital accumulation via learning by doing. The model describes dynamic interactions among agricultural and industrial production, rural and urban amenities, distribution of production factors and preferences with endogenous capital and human capital accumulation. We show that the dynamic system may have either a single or multiple equilibrium points, depending upon returns to scale in the two sectors. We also examined effects of changes in some parameters.

KEY WORDS

two-sector model, agricultural sector, industrial sector, physical capital accumulation, human capital accumulation, rural and urban amenities

CLASSIFICATION

JEL: R11, R14

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INTRODUCTION

It is important to study urbanization with labour migration as in many less developed countries an important part of the population is still devoted to agriculture. In economies like India, China and some African countries, agricultural population shares a high percentage of the total population. On the other hand, industrialization and human capital accumulation are altering dramatically the labour distribution between agricultural and non-agricultural sectors in many developing economies. It is well known that the Harris-Todaro framework has played the role of a key model in analyzing industrialization with labour migration between the urban and rural areas[1].The Harris-Todaro model attempts to explain persistent rural-urban migration despite the high unemployment rates in cities, especially in developing economies. In this model, the formal-sector wage is fixed at a level far above the agricultural wage and the migration decision is based on expected earnings. To maintain the presumed equalization of expected earnings, some urban residents are unemployed as migration is costless. The change in the probability of formal employment is the principal mechanism that restores migration equilibrium in response to exogenous changes such as technical progress and job or wage growth in the city. As pointed out by Brueckner and Zenou [1], there are many other factors that limit urban growth. For instance, a rise in the city population raises the urban living cost, mostly likely through the land markets, which limits urban growth. As the urban population rises in response to positive shocks, land prices tend to rise, lowering the utility levels of all urban residents. The gap between rural utility and the expected utility of an urban resident is closed up by migration. This study examines migration equilibrium without unemployment. Although we still assume equalization of utility levels between the urban and rural areas, we analyze differences in urban and rural living conditions by assuming that urban and rural areas offer not only different wage rates but also different levels of amenity. This study shows how productivities, land and amenity interact to determine labour distribution between the urban and rural areas in the long term. By taking account of endogenous amenity and land and human capital accumulation, we try to offer an alternative approach to the economy described by the Harris-Todaro model.

Another important issue related to economic structural change is dynamics of human capital and technological change. It is well known that it is difficult to introduce both human capital and physical capital accumulation as endogenous variables into the Harris-Todaro framework because of analytical intractability. For instance, Matsuyama [2] examines how agricultural productivity influences economic growth and the process of industrialization. The model shows that the effect of agricultural productivity on growth is crucially dependent on openness to trade. Nevertheless, Matsuyama's analysis relies on the assumption that agriculture is backward and no technological progress will take place in the sector. The growth process is driven solely by learning by doing in manufacturing. Nevertheless, the assumption that human capital accumulation is negligible through learning by doing in agricultural sector is not realistic[2]. This study takes account of human capital accumulation both in industrial and agricultural sectors. Multiple equilibrium points exist when the two sectors exhibit increasing and decreasing returns to scale. Although our model is constructed in dynamics, because of the nature of the problem, it is difficult to carry out a complete dynamic analysis. This study is mainly concerned with issues of existence of equilibrium and comparative statics analysis.

The paper constructs a two-sector growth model with endogenous human capital and physical capital accumulation. The model tries to provide some insights into processes of industrialization and urbanization with labour migration. Although the paper studies issues similar to those addressed by the Harris-Todaro model and its various extensions, we deviate the traditional approach by proposing an alternative approach to household behaviour. The equilibrium mechanism of labour migration is expressed by equalizing utility levels in the urban and rural areas. Different from the Harris-Todaro approach, we use the concept of amenity to reflect living and work condition differences between the urban and rural areas. The wage rates differ between the industrial and agricultural sectors because the urban and rural areas offer different levels of amenity and land rent. It should be noted that this paper is an extension of a model proposed by Zhang [3; Ch. 6]. The main difference between this model and Zhang's model is that this study introduces differences in amenity between urban and rural areas, while Zhang's does not take account of possible differences in amenity in different professions and economic geography. The paper is organized as follows. Section 2 defines the two-sector growth model with physical and human capital accumulation. Section 3 provides the process to determine all the variables and demonstrate existence of equilibrium when the parameter values are specified. Section 4 examines effects of changes in the total productivity, the population, and propensity on the levels of physical and human capital and economic structure. Section 5 concludes the study. Appendix A.1 proves the process of finding equilibrium in Section 3. Appendix A.2 shows how to express the dynamics of the economic variables in a three differential equations system.

ECONOMIC GROWTH WITH PHYSICAL AND HUMAN CAPITAL ACCUMULATION

Similar to Harris and Todaro [4] and Irz and Roe [5], we consider an economic system consisting of agricultural and industrial sectors. The agricultural sector produces goods such as corn, rice and vegetables, which are only for consumption. The industrial sector produces commodities for investment and consumption. Industrial commodity is selected to serve as numeriare. It is assumed that labour force, land and capital are always fully employed[3]. The population is assumed to be homogenous in the sense that their preference and skill structures are identical. This implies that people can costlessly move from countryside to city, and vice versa. A person is free to choose his residential location. We assume that any person chooses the same area where he works and lives. Each area has fixed land. Land quality, climates, and environment are homogenous within each area, but they may vary between the areas. We neglect transportation cost of commodities[4]. As become evident later on, although it is conceptually not difficult to introduce transportation cost function and to provide balance conditions for demand and supply and for price equalization conditions with transportation cost, the problem will become analytically too complicated. The assumption of zero transportation cost of commodities implies price equality for the commodity over space. Nevertheless, as amenity and land are immobile, wage rates and land rent vary between the areas.

BEHAVIOUR OF PRODUCTION SECTORS

We denote $K(t)$, $r(t)$ and $p(t)$ the total capital, the rate of interest and price of agricultural commodity, respectively. We define the following indexes and variables

a, i – subscripts denoting agriculture and industry,
N – the total fixed labour force of the economy,
L_i and L – the fixed urban and rural areas,
$N_j(t)$ and $K_j(t)$ – the labour force and capital stocks employed by sector j ($j = a, i$) at time t,
$L_a(t)$ – the land employed by the agricultural sector,
$F_j(t)$ and $C_j(t)$ – sector j's output and consumption levels of product j, and
$w_j(t)$ – sector j's wage rate.

We assume that production processes can be described by some aggregate production functions. We assume that agricultural production is a process of combining land, labour force and capital. For simplicity, the production function of the agricultural sector is specified as follows

$$F_a(t) = A_a K_a^{\alpha_a} (H^{m_a/\beta_a} N_a)^{\beta_a} L_a^{\varsigma}, \;\; m_a \geq 0, \;\; \alpha_a + \beta_a + \varsigma = 1, \;\; \alpha_a, \beta_a, \varsigma > 0. \qquad (1)$$

Here, the term $H^{m_a/\beta_a} \cdot N_a$ is the qualified labour input. The parameter m_a/β_a describes how effectively the agricultural sector utilizes human capital. The marginal conditions for the agricultural sector are given by

$$r + \delta_k = \frac{\alpha_a p F_a}{K_a}, \;\; w_a = \frac{\beta_a p F_a}{N_a}, \;\; R_a = \frac{\varsigma p F_a}{L_a}, \qquad (2)$$

where δ_k is depreciation rate of physical capital.

The industrial production is a process of combining labour force and capital. The land use by the industrial sector is omitted[5]. The production function of the industrial sector is specified as follows

$$F_i(t) = A_i K_i^{\alpha_i} (H^{m_i/\beta_i} N_i)^{\beta_i}, \;\; m_i \geq 0, \;\; \alpha_i + \beta_i = 1, \;\; \alpha_i, \beta_i > 0. \qquad (3)$$

The marginal conditions for the industrial sector are given by

$$r + \delta_k = \frac{\alpha_i F_i}{K_i}, \;\; w_i = \frac{\beta_i F_i}{N_i}. \qquad (4)$$

We described behaviour of the production sectors.

CONSUMER BEHAVIOUR

Each worker may get income from land ownership, wealth ownership and wages. To simplify the model, we accept the assumption of “equally shared landownership” which means that the income of land rent is equally distributed among the population. The total land revenue is given by $R_i(t) \cdot L_i + R_a(t)\, L$, where $R_i(t)$ and $R_a(t)$ are the land rents in the city and the rural area, respectively. Each consumer obtains the following land revenue

$$\bar{r}(t) \equiv \frac{R_i(t) L_i + R_a(t) L}{N}. \qquad (5)$$

This study uses the approach to consumers’ behaviour proposed by Zhang in the early 1990s [3, 6]. This approach makes it possible to solve many national, international, urban, and interregional economic problems, such as growth problems with heterogeneous households, multi-sectors, and preference changes, which are analytically intractable by the traditional approaches in economics. Let $\bar{k}_j(t)$ stand for the per capita wealth (excluding land) owned by the typical household j Each household of area j obtains income

$$y_j(t) = r(t)\bar{k}_j(t) + w_j(t) + \bar{r}(t), \;\; j = i, a,$$

from the interest payment, $r\bar{k}_j$, and the wage payment, w_j and the land revenue, $\bar{r}$. We call y_i the current income in the sense that it comes from consumers’ wages and current earnings from ownership of wealth. The sum of income that consumers are using for consuming, saving, or transferring are not necessarily equal to the current income because consumers can sell wealth to pay, for instance, the current consumption if the current income is not sufficient for buying food and touring the country. Retired people may live not only on the interest payment but also have to spend some of their wealth. The total value of the wealth that consumer j can sell to purchase goods and to save is equal to $p_i(t)\; \bar{k}_j(t)$ with $p_i(t) = 1$ at any t Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income is then equal to

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t). \tag{6}$$

The disposable income is used for saving and consumption. It should be noted that the value, $\bar{k}_j(t)$ (i.e., $p_i(t)\bar{k}_j(t)$), in the above equation is a flow variable. Under the assumption that selling wealth can be conducted instantaneously without any transaction cost, we may consider $\bar{k}_j(t)$ as the amount of the income that the consumer obtains at time t by selling all of his wealth. Hence, at time t the consumer has the total amount of income equaling $\hat{y}_j(t)$ to distribute between consuming and saving. It should also be remarked that in the growth literature, for instance, in the Solow model, the saving is out of the current income, $y_i(t)$ while in this study the saving is out of the disposable income. This approach is discussed at length elsewhere [3, 6][6].

At each point of time, a consumer distributes the total available budget among housing, $l_j(t)$ saving, $s_j(t)$ consumption of agricultural goods, $c_{ja}(t)$ and consumption of industrial goods, $c_{ji}(t)$. The budget constraint is given by

$$R_j l_j + c_{ji} + p c_{ja} + s_j = \hat{y}_j. \tag{7}$$

Furthermore, at each point of time, consumers have four variables to decide. A consumer decides how much to consume housing, industrial and agricultural goods, and how much to save. Equation (7) means that consumption and savings exhaust the consumers' disposable personal income.

We assume that utility level, $U_j(t)$, that the consumer j obtains is dependent on $l_j(t)$, $c_j(t)$, $c_{ja}(t)$ and $s_j(t)$. The utility level of the typical consumer in region j is represented by

$$U_j(t) = \theta_j(t) l_j^{\eta_0}(t) c_j^{\xi_0}(t) c_{ja}^{\mu_0}(t) s_j^{\lambda_0}(t), \quad \eta_0, \xi_0, \mu_0, \lambda_0 > 0, \quad j = a, i, \tag{8}$$

in which η_0, ξ_0, μ_0 and λ_0 are a typical person's elasticity of utility with regard to lot size, industrial goods, agricultural goods, and savings in area j, respectively. We call η_0, ξ_0, μ_0 and λ_0 the propensities to consume lot size, industrial goods, agricultural goods, and to hold wealth (save), respectively. In (8), $\theta_a(t)$ and $\theta_i(t)$ are respectively called the rural and urban amenity levels. Amenities are affected by, for instance, infrastructures, professional. In this study, we assume that amenity is affected by production and consumption activities. We specify θ_j as follows

$$\theta_j(t) = \bar{\theta}_j N_j^{d_j}(t), \quad j = a, i, \tag{9}$$

where $\bar{\theta}_j$ (>0) and d_j are parameters. We do not specify sign of d_j as the population may have either positive or negative effects on the attractiveness of a location[7]. Maximizing $U_j(t)$ subject to the budget constraints yields

$$l_j(t) R_j(t) = \eta \hat{y}_j(t), \quad c_j(t) = \xi \hat{y}_j(t), \quad p(t) c_{ja}(t) = \mu \hat{y}_j(t), \quad s_j(t) = \lambda \hat{y}_j(t). \tag{10}$$

where

$$\eta \equiv \rho \eta_0, \quad \xi \equiv \rho \xi_0, \quad \mu \equiv \rho \mu_0, \quad \lambda \equiv \rho \lambda_0, \quad \rho \equiv (\eta_0 + \xi_0 + \mu_0 + \lambda_0)^{-1}.$$

As shown in [3], the saving behaviour of the approach in this study is similar to these implied by the Keynesian consumption function and permanent income hypotheses, which are empirically more valid than the assumptions in the Solow model with a constant saving rate or the Ramsey model[8]. It should be remarked that the saving, $s(t)$ defined in this study is different from the saving in the Solow model. It can be shown that the approach to consumers' saving behaviour in this study can generate the same behaviour as in the Solow model or the Ramsey model when the propensity to save, λ, is assumed to be related to the wealth and income[9].

According to the definitions of $s_j(t)$ the wealth accumulation of the representative household in area j is given by

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \tag{11}$$

As households are assumed to be freely mobile between the two areas, the utility level of people should be equal, irrespective of in which area they live, i.e.

$$U_i(t) = U_a(t). \tag{12}$$

We neglect possible costs for migration. In reality, even to change a house in a small town costs. Although it is not difficult to introduce migration costs into the model, it will become far more difficult to explicitly get analytical results. In this study, instead of wage equalization (which is often used as the equilibrium mechanism of population distribution), we assume that consumers obtain the same level of utility in different professions as the equilibrium mechanism of population distribution between the professions. Although the condition of utility equalization is often used in the literature of urban economics, the assumption of utility equalization is not often used in the literature of economic dynamics as the temporary equilibrium condition of population distribution. It is argued that this assumption is more reasonable than the assumption of wage equalization.

The total capital stock employed by the production sectors is equal to the total wealth owned by all the regions. That is

$$K(t) = K_a(t) + K_i(t) = \bar{k}_a(t)N_a(t) + \bar{k}_i(t)N_i(t). \tag{13}$$

The national demand for and supply of agricultural goods is equal. That is

$$c_{aa}(t)N_a(t) + c_{ia}(t)N_i(t) = F_a(t) \tag{14}$$

The national production of industrial goods is equal to the national consumption and national net saving. That is

$$C(t)+S(t) - K(t)+\delta_k K(t)=F_i(t), \tag{15}$$

where

$$C(t) \equiv c_a(t)N_a(t) + c_i(t)N_i(t),\ S(t) \equiv s_a(t)N_a(t) + s_i(t)N_i(t).$$

The assumption that labour force and land are fully employed is represented by

$$N_a(t) + N_i(t) = N,\ l_i(t)N_i(t) = L_i,\ L_a(t) + l_a(t)N_a(t) = L. \tag{16}$$

HUMAN CAPITAL ACCUMULATION

We assume that there are two sources of improving human capital, through learning by doing[10]. Arrow [7] first introduced learning by doing into growth theory. We specify the following dynamics[11]

$$\dot{H} = \frac{\tau_a N_a F_a}{NH^{\delta_a}} + \frac{\tau_i N_i F_i}{NH^{\delta_i}} - \delta_h H, \tag{17}$$

where τ_j, ε_j and δ_h are parameters. The term $\delta_h \cdot H$ describes depreciation of human capital, where δ_h is the depreciation rate of human capital. We interpret $\tau_j \cdot N_j \cdot F_j / NH^{\varepsilon_j}$ as effects of learning by doing of each worker in sector j upon accumulation of human capital. The contribution of the production sector to human capital improvement is positively related to its production scale, F_j, and is dependent on the level of human capital. The term H^{ε_j} takes account of returns to scale effects in human capital accumulation. The case of $\varepsilon_j > (<) 0$ implies that as human capital is increased it is more difficult (easier) to further improve the level of human capital. The term, N_j/N measures sector j's relative contribution to the improvement of human capital.

We have thus established the economic dynamics with endogenous economic structure, physical capital and human capital. We now examine dynamic properties of the system.

ECONOMIC EQUILIBRIUM

This section shows that the dynamic system may have either a unique or none or multiple equilibrium points. Since a complete dynamic analysis system is too complicated, we are only concerned with existence of equilibrium[12]. Before stating the main analytical results, we introduce two parameters

$$x_a \equiv \frac{\alpha_a m_i}{\beta_i} + m_a - \varepsilon_a - 1, \;\; x_i \equiv \frac{m_i}{\beta_i} - \varepsilon_i - 1.$$

The following proposition is proved in Appendix A1.

PROPOSITION

The equilibrium values of r and N_i are uniquely given by equations (A11) and (A17). For $0 < N_i < N$ and $r > 0$ if $x_a < 0$ and $x_i < 0$ (or $x_a > 0$ and $x_i > 0$), the system has a unique equilibrium; and if $x_a < 0$ and $x_i > 0$ ($x_a > 0$ and $x_i < 0$), the system may have none, one, or two equilibrium points. For a positive value of H determined by (A20), the equilibrium values of all the other variables are uniquely determined by the following procedure: $N_a = N - N_i \rightarrow \bar{k}_j$ (j = a, i) by (A19) $\rightarrow K_i$ by (A18) $\rightarrow R_j$ by (A14) $\rightarrow w_i$ by (A11) $\rightarrow w_a$ by (A12) $\rightarrow \bar{r}$ by (A10) $\rightarrow \hat{y}_j = \bar{k}_j / \lambda \rightarrow K$ by (A9) $\rightarrow K_a$ by (A8) $\rightarrow p$ by (A2) $\rightarrow l_i = L_i/N_i \rightarrow l_a$ by (10) $\rightarrow c_a$, c_{aa} and s_a by (10) $\rightarrow c_i$, c_{ia} and s_i by (10) $\rightarrow F_i$ by (3) $\rightarrow F_a$ by (1).

By the definitions of x_a and x_i, we interpret x_a and x_i as measurements of returns to scale of the agricultural and industrial sectors in the dynamic system, respectively. When $x_j < (>)\ 0$ we say that sector j displays decreasing (increasing) returns to scale in the dynamic economy. The above proposition tells us that if the sectors both display decreasing (increasing) returns, the dynamic system has a unique equilibrium; if one sector displays decreasing (increasing) returns and the other sector exhibits increasing (decreasing), the system may have none, one, or two equilibrium points. As shown in Appendix A2, it is difficult to explicitly judge stability properties of the dynamic system. Nevertheless, if the urban and rural areas have the same level of constant amenity, then the dynamic analysis becomes much easier[13]. The following corollary is proved in [3].

COROLLARY

Assume that the urban and rural areas have the same level of constant amenity, that is, $\theta_i = \theta_a$. Then, if $x_a < 0$ and $x_i < 0$ (or $x_a > 0$ and $x_i > 0$), the system has a unique stable (unstable) equilibrium point; and if $x_a < 0$ and $x_i > 0$ ($x_a > 0$ and $x_i < 0$), the system may have none, one, or two equilibrium points. When the system has two equilibrium points, the one with higher value of H is stable and the other one is unstable.

The assumption of $\theta_i = \theta_a$ and the same level of land rents for different types of land use imply that the same level and consumption pattern of households, irrespective of their professions and location. Under this strict requirement, we can explicitly determine the dynamic properties of the model. Nevertheless, like in the Harris-Todaro model, this study is concerned with effects of rural and urban production and living condition differences upon labour migration. As shown in Appendix A2, it is difficult to analyze dynamic properties of the model. For illustration, we specify values of the parameters and simulate the model to examine the behaviour of the economic system. We specify the parameters as follows

$$\alpha_i = 0{,}45,\ \alpha_a = 0{,}25,\ \beta_a = 0{,}35,\ A_i = 1{,}1,\ A_a = 0{,}9,\ d_a = 0,\ d_i = -0{,}05,\ \bar{\theta}_i = 4,\ \bar{\theta}_a = 5,$$
$$\eta_0 = 0{,}07,\ \xi_0 = 0{,}1,\ \mu_0 = 0{,}05,\ \lambda_0 = 0{,}7,\ \tau_a = 0{,}01,\ \tau_i = 0{,}05,\ \varepsilon_a = 0{,}5,\ \varepsilon_i = 0{,}05,$$
$$m_a = 0{,}3,\ m_i = 0{,}7,\ N = 10,\ L_i = 1,\ L = 10,\ \delta_k = 0{,}03,\ \delta_h = 0{,}1. \qquad (18)$$

The capital shares in the industrial and agricultural sectors, α_i and α_i are equal to 0,45 and 0,25, respectively. This implies that the industrial sector is relatively capital-intensive compared with agriculture[14]. The total productivity levels of the industrial and agricultural sectors, A_i and A_a are 1,1 and 0,9, respectively. The level of the industrial sector is higher than the agricultural sector. The amenity coefficients of the urban and rural areas, $\bar{\theta}_i$ and $\bar{\theta}_a$ are fixed at 4 and 5, respectively. As shown in the proposition, what matters is the ratio $\bar{\theta}_i / \bar{\theta}_a$, rather than their absolute values. The lower the ratio is, the more attractive the rural area becomes, with all other conditions equal. For simplicity, we assume that the rural amenity is constant and the urban amenity falls as the city's population rises[15]. The propensity to save, λ_0 is 0,7. The specified values of η_0, ξ_0 and μ_0 imply that the ratio between the expenditures on the housing and agricultural goods is 1,4 and the ratio between the expenditures on the industrial goods and agricultural goods is 2. The total population is 10 and the rural territory size is 10 times of the urban territory size. The conditions $\varepsilon_a = 0{,}5$ and $\varepsilon_i = 0{,}05$ mean respectively that the learning by producing exhibits decreasing effects in human capital; the agricultural sector's decreasing effect is much stronger than the industrial sector's.

Under (18) we have $x_a = -0{,}822$ and $x_i = -0{,}223$ This implies that the agricultural sector's learning by doing exhibits decreasing returns and the industrial sector increasing returns. As shown in the proposition, the system may have two equilibrium points. From the proposition we know that the variables, r, N_i, N_a, L_a, l_a and l_i are determined, independent of the two variables H and K_i. This implies that when the system has two equilibrium points, the rate of interest, the labour distribution and land-use distribution are equal at the two points[16]. The variable N_i is determined by equation (A17), $\Phi_N(N_i) = 0$. Figure 1 shows that the equation has a unique solution.

We uniquely determine r, N_i, N_a, L_a, l_a and l_i as in Table 1. We see that most of the labour force is located in the city. The farmer's lot size is much larger than the urban worker's lot size The variable H is determined by equation (A20), $\Phi_H(H) = 0$. Figure 2 shows that the equation

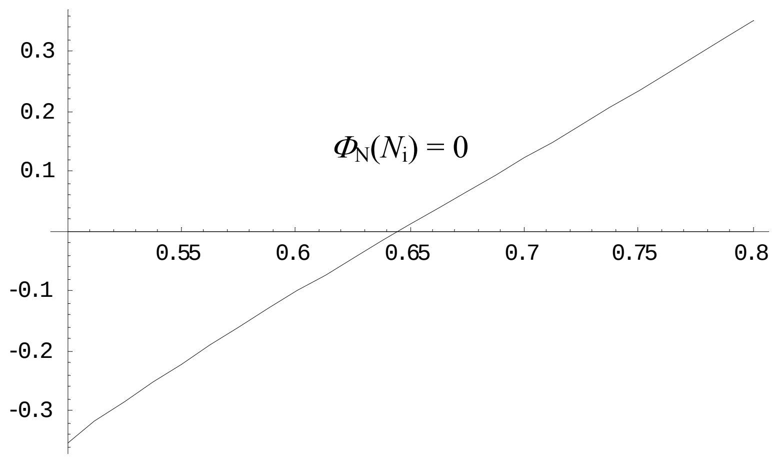

Figure 1. The unique labour distribution.

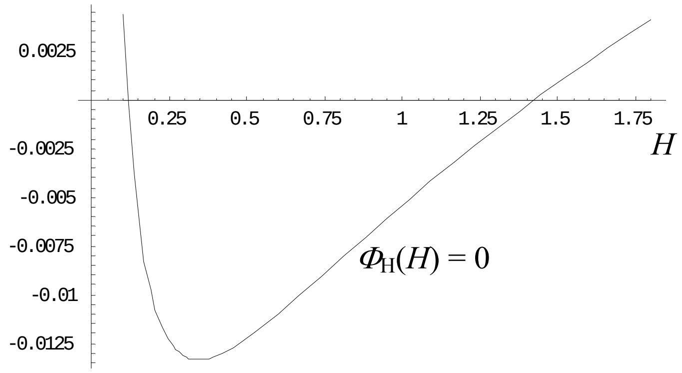

Figure 2. The existence of two equilibrium points of human capital.

has two solutions: H_1 = 0,115 and H_2 = 1,427. We denote the two equilibrium points using subscripts 1 and 2. We call the two equilibrium points as advanced equilibrium (AE) and underdeveloped equilibrium (UE). Following the proposition, we determine the equilibrium values of the other variables, which are summarized as in Table 1.

Table 1. The variables' values at the two equilibrium points, H_1 and H_2.

r	N_i	N_a	L_a	l_a	l_i
0,065	0,645	0,355	8,347	4,655	1,551

Variable	Equilibrium point		Variable	Equilibrium point		Variable	Equilibrium point	
	H_1	H_2		H_1	H_2		H_1	H_2
F	0,211	8,504	F_a	0,504	2,388	$\bar{k}_a$	0,161	3,946
H	0,115	1,427	F_i	0,174	4,280	$\bar{k}_i$	0,223	5,473
K	1,001	24,761	w_a	0,071	1,743	c_{aa}	0,016	0,159
p	0,072	1,769	w_i	0,149	3,651	c_a	0,023	0,564
$\bar{r}$	0,130	3,184	R_a	0,003	0,085	c_{ia}	0,021	0,221
$\bar{K}_a$	0,188	4,623	R_i	0,014	0,353	c_i	0,016	0,391
$\bar{K}_i$	0,820	20,138						

We see that the difference in the levels of human capital at the two equilibrium points is very large. The level of the human capital at the AE is more than 12 times higher than that at the UE. The ratio between the national output levels, F (= $F_i + p \cdot F_a$), is 42 times. The price of the agricultural goods, land rent, and the wage rate at the AE are all much higher than the corresponding variables at the UE. The output levels of the two sectors, the per capita wealth levels of the rural and urban residents, and the per capita consumption levels of the two products by the rural and urban residents at the AE are all much higher than the corresponding variables at the UE.

In the literature of economic development, it is well known that there may be multiple equilibrium points for the same type of economy when market imperfections or endogenous human capital are introduced into economic dynamics. This implies, for instance, that two seemingly identical regions may follow radically different development paths, one leading to prosperity, the other to stagnation. Taiwan and Mainland China may provide a proper case for this result. Although they had similar backgrounds in terms of cultural heritage, values, and initial human capital, Taiwan and Mainland China had experienced totally different paths of industrialization during the period 1950-1980 – the former rapidly moved to the high equilibrium point, while the latter cycled around the low equilibrium point. It should be remarked that Canning [8] proposes a two-sector model with increasing returns to scale in the industrial sector and diminishing returns in agriculture. The model demonstrates that increasing demand for food coupled with diminishing returns in agriculture may not be a barrier to economic growth[17]. Canning's model shows that the growth of the economy may be unlimited, despite ever increasing demand for agricultural procedure and in the absence of technical progress, if the increasing demands in the capital goods industry are sufficient to outweigh the diminishing returns to capital in agriculture. The equilibrium point with the higher level of human capital in our model explains what the Canning model predicts. It should be remarked that the concerns of classical economists, such as Ricardo, about capital accumulation with agriculture and industry can be explained by the case of the decreasing returns to scale in the two sectors.

CHANGES IN THE PRODUCTIVITY LEVEL, THE POPULATION, AND THE PROPENSITY TO SAVE

We now examine how the parameters affect the economic structure and labour distribution. First, we examine the case that all the parameters, except the productivity of the industrial sector A_i, are the same as in (18). We increase the productivity level A_i from 1,1 to 1,15. We introduce a symbol $\bar{\Delta}$ to stand for the change rate of the equilibrium value of a variable in percentage due to the change in a parameter value from. For instance, with regard to a variable x_j, assuming the change of a parameter A_i from its current value A_{i0} (which equals 1,1 in this case) to the new value A_{i1} (equal to 1,15), we have

$$\bar{\Delta} x_j \equiv \frac{x_j(A_{i1}) - x_j(A_{i0})}{x_j(A_{i0})} \times 100.$$

As A_i rises from 1,1 to 1,15 the variables, r, N_i, N_a, L_a, l_a and l_i are not affected. Figure 3 shows how the two solutions of H are affected. The equilibrium values of the other variables are listed in Table 2. We can see that an increase in the industrial sector's total productivity has the opposite effects upon the variables at the AE and the UE. The equilibrium values at the UE are increased and the equilibrium values at the AE are reduced. This implies that for the economy under consideration, as the total productivity is improved, in order to maintain the system at equilibrium the lower equilibrium point is improved and the higher equilibrium point is lowered. Intuitively, it is easy to interpret the effects upon the UE as this point is characterized of decreasing returns to scale. An increase in the productivity will improve the economic performance of the economy. To interpret the effects upon the AE, first we note that this point is characterized of increasing returns to scale. Although we fail to prove its stability properties, this point is seemingly unstable. This implies that if the system is located near the AE, it has possibility of unlimited growth as the system will rarely remind at unstable equilibrium in the long term. If the equilibrium point is lowered, it is easier for the economy to sustain economic growth in the long term[18].

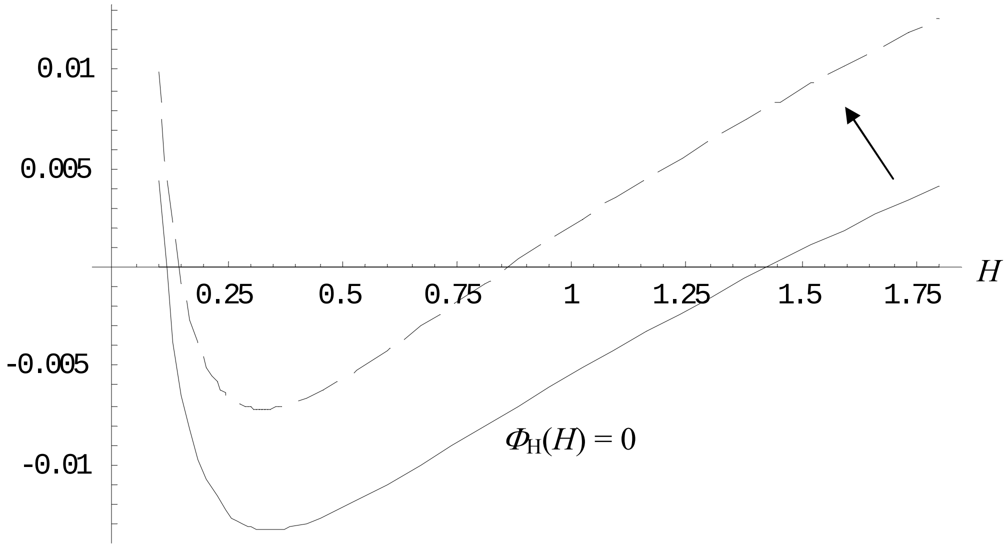

Figure 3. Shifts of the two equilibrium points as A_i rises.

Table 2. The effects as the industrial sector's total productivity rises. Values of variables are given at two equilibrium points, H_1 and H_2.

Variable	Equilibrium point		Variable	Equilibrium point		Variable	Equilibrium point	
	H_1	H_2		H_1	H_2		H_1	H_2
$\bar{\Delta}F$	49,971	–50,099	$\bar{\Delta}F_a$	17,724	–25,932	$\bar{\Delta}\bar{k}_a$	45,524	–42,932
$\bar{\Delta}H$	26,020	–39,936	$\bar{\Delta}F_i$	45,524	–42,932	$\bar{\Delta}\bar{k}_i$	45,524	–42,932
$\bar{\Delta}K$	45,524	–42,932	$\bar{\Delta}w_a$	45,524	–42,932	$\bar{\Delta}c_{aa}$	0	0
$\bar{\Delta}p$	45,524	–42,932	$\bar{\Delta}w_i$	45,524	–42,932	$\bar{\Delta}c_a$	45,524	–42,932
$\bar{\Delta}\bar{r}$	45,524	–42,932	$\bar{\Delta}R_a$	45,524	–42,932	$\bar{\Delta}c_{ia}$	0	0
$\bar{\Delta}\bar{K}_a$	45,524	–42,932	$\bar{\Delta}R_i$	45,524	–42,932	$\bar{\Delta}c_i$	45,524	–42,932
$\bar{\Delta}\bar{K}_i$	45,524	–42,932						

We also analysed the case when A_a: 0,9 → 1. It can be shown that the variables r, N_i, N_a, L_a, l_a and l_i are not affected and the two solutions of H are changed similarly to Figure 3. The change directions in the variables are the same as in Table 2, except that the per capita consumption levels of the farmers are affected but the consumption levels of the urban workers are not affected. As we increase τ_i: 0,05 → 0,055 the variables, r, N_i, N_a, L_a, l_a and l_i are not affected and the two solutions of H are changed similarly to Figure 3. The change directions in the variables are the same as in Table 2. We now allow the population to rise as follows: N: 1,0 → 1,1. It is demonstrated that the two solutions of H are changed similarly to Figure 3. The level of human capital at UE is increased and the level at the AE is reduced. The changes in the equilibrium values of the variables are given in Table 3. As the total population rises, the rate of interest is not affected. The urban population rises by 9,6 % and the rural population rises by 10,8 %. The residential lot sizes in the rural and urban area fall respectively by 9,4 % and 8,7 %. The level of human capital at the UE rises by 41 % and the level of human capital at the AE falls by 44 %. We see that an increase in the population has the opposite effects upon the variables at the AE and the UE, except for the per capital levels of consumption of the agricultural goods which are reduced at the both equilibrium points. The equilibrium values at the UE are increased and the equilibrium values at the AE are reduced. A larger population benefits the long-term economic growth. As the population is increased, the UE is improved and the AE is lowered.

Table 3. The effects as the population rises.

$\bar{\Delta} r$	$\bar{\Delta} N_i$	$\bar{\Delta} N_a$	$\bar{\Delta} L_a$	$\bar{\Delta} l_a$	$\bar{\Delta} l_i$
0,065	0,645	0,355	8,347	4,655	1,551

Variable	Equilibrium point		Variable	Equilibrium point		Variable	Equilibrium point	
	H_1	H_2		H_1	H_2		H_1	H_2
$\bar{\Delta} F$	78,649	–54,263	$\bar{\Delta} F_a$	31,073	–25,852	$\bar{\Delta} \bar{k}_a$	53,750	–52,419
$\bar{\Delta} H$	40,945	–43,918	$\bar{\Delta} F_i$	69,563	–47,526	$\bar{\Delta} \bar{k}_i$	54,483	–52,192
$\bar{\Delta} K$	69,583	–47,526	$\bar{\Delta} w_a$	53,023	–52,644	$\bar{\Delta} c_{aa}$	–9,326	–9,326
$\bar{\Delta} p$	69,583	–47,526	$\bar{\Delta} w_i$	54,775	–52,102	$\bar{\Delta} c_a$	53,750	–52,419
$\bar{\Delta} \bar{r}$	54,148	–52,296	$\bar{\Delta} R_a$	69,563	–47,484	$\bar{\Delta} c_{ia}$	–8,893	–8,894
$\bar{\Delta} \bar{K}_a$	69,563	–47,526	$\bar{\Delta} R_i$	69,563	–47,626	$\bar{\Delta} c_i$	54,483	–52,419
$\bar{\Delta} \bar{K}_i$	69,563	–47,526						

An important issue in growth theory is related to interdependence between the propensity to save and national wealth. The study of individual thrift and national wealth has long been important in economics because national saving is the source of the supply of capital, a main factor of production affecting the productivity of labour. Thrift had traditionally been regarded as a virtuous, socially beneficial act. Admit Smith argued that capital is increased by parsimony and diminished by prodigality. He believed that parsimony, and not industry, is the immediate cause of the increase in capital. Smith said that prodigals are public enemies. This belief was strongly challenged by Keynes in the General Theory. He suggested that saving is potentially disruptive to the economy and harmful to social welfare. High propensity to save may reduce consumption, without systematically and automatically giving rise to an offsetting expansion in investment. This might thus cause demand to fall lower than proper level and hence output and employment lower than the capacity of the economy. We show that in economies with returns to scale the impact of the propensity to hold wealth are situation-dependent. An increase in the propensity to save may either increase or reduce the national wealth, depending on the current situations of the system. This implies that both Smith and Keynes are right under some situations and wrong under others.

We increase the propensity to save as follows, λ_0: 0,7 $\rightarrow$ 0,73. The two solutions of H are changed similarly to Figure 3. The changes in the equilibrium values of the variables are listed in Table 4. As the propensity to save rises, the rate of interest falls. The urban population rises and the rural population falls. The land for agricultural use is increased. The lot sizes in the urban and rural areas are reduced. The levels of human capital and national output are reduced at the AE and increased at the UE.

Similar to the impact of an increase in the industrial sector's total productivity, an increase in the propensity to save has the opposite effects upon the variables at the AE and the UE, except the variables, p and $\bar{r}$ which are increased, and c_{aa} and c_{ia} which are reduced at the both equilibrium points. To see how c_{aa} and c_{ia} are reduced at the both equilibrium points, we note that as the propensity to save rises, the propensities to consume lot size, industrial goods, and agricultural goods fall relatively. The falls in the propensities tend to reduce the lot size, the consumption levels of the industrial agricultural goods and affect the prices of these goods. On the other hand, the changes in the incomes also affect the consumption levels of these variables and their prices. The net effects upon the consumption levels of the agricultural goods are negative at the both equilibrium points[19].

Table 4. The effects as the propensity to save rises.

$\bar{\Delta} r$	$\bar{\Delta} N_i$	$\bar{\Delta} N_a$	$\bar{\Delta} L_a$	$\bar{\Delta} l_a$	$\bar{\Delta} l_i$
–5,153	1,254	–2,276	0,799	–1,797	–1,238

Variable	**Equilibrium point**		**Variable**	**Equilibrium point**		**Variable**	**Equilibrium point**	
	H_1	H_2		H_1	H_2		H_1	H_2
$\bar{\Delta} F$	20,050	–36,361	$\bar{\Delta} F_a$	7,821	–16,760	$\bar{\Delta} \bar{k}_a$	19,622	–29,763
$\bar{\Delta} H$	10,633	–27,176	$\bar{\Delta} F_i$	18,613	–30,355	$\bar{\Delta} \bar{k}_i$	18,325	–30,524
$\bar{\Delta} K$	22,783	–27,906	$\bar{\Delta} w_a$	20,480	–29,259	$\bar{\Delta} c_{aa}$	–2,575	–2,575
$\bar{\Delta} p$	17,737	30,869	$\bar{\Delta} w_i$	17,145	–31,217	$\bar{\Delta} c_a$	14,706	–32,649
$\bar{\Delta} \bar{r}$	17,737	30,869	$\bar{\Delta} R_a$	16,805	–31,417	$\bar{\Delta} c_{ia}$	–3,631	–3,631
$\bar{\Delta} \bar{K}_a$	22,045	–28,340	$\bar{\Delta} R_i$	14,885	–32,544	$\bar{\Delta} c_i$	13,462	–33,379
$\bar{\Delta} \bar{K}_i$	22,953	–27,807						

CONCLUDING REMARKS

This paper examines issues related to urbanization with labour migration. The model is influenced by the Harris-Todaro model. Main departures from the traditional approach are that this paper uses an alternative approach to consumer and introduces human capital accumulation via learning by doing. The Harris-Todaro assumes unemployment of labour force in the urban area, while our study assumes full employment in the city as well. We explain differences in living conditions and wages between the urban and rural areas by introducing endogenous amenities. As amenities and technology vary between the city and rural area, the wage rates, housing rents and consumption levels are different between the city and rural area are different. The economic system consists of one production sector and one education sector. The model describes dynamic interactions among agricultural and industrial production, rural and urban amenities, distribution of production factors and preferences with endogenous capital and human capital accumulation. We show that the dynamic system may have either a single or multiple equilibrium points, depending on returns to scale parameters. We also examined effects of changes in some parameters. We get some insights into important issues related to relationships between living conditions and population growth, and the effects of propensity to save. For instance, we showed that in economies with returns to scale the impact of the propensity to hold wealth are situation-dependent. An increase in the propensity to save may either increase or reduce the national wealth, depending on the current situations of the system. This implies that both Smith and Keynes are right under some situations and wrong under others. Finally, it should be remarked that our comparative statics analysis is based on the specified parameter values. It is not difficult to see that effects of changes in any parameters are situation-dependent in the economy. We may extend the model in some directions. For instance, we may introduce some kind of government intervention in education into the model. It is also desirable to treat leisure time as an endogenous variable.

APPENDICES

A1: PROVING THE PROPOSITION

By equations (11) and (17) at equilibrium we have $s_j = \bar{k}_j$, $j = a, i$.

$$\frac{\tau_a N_a F_a}{NH^{\delta_a}} + \frac{\tau_i N_i F_i}{NH^{\delta_i}} = \delta_h H . \quad \text{(A1)}$$

From equations (2) and (4) we have

$$p = \frac{\alpha_i K_a F_i}{\alpha_a K_i F_a} \quad \text{(A2)}$$

Substitute $L_a = \varsigma \cdot p \cdot F_a / R_a$ in (2) and lj$\cdot R_j = \eta \cdot \hat{y}_j$ in (10) into the land constraints (16)

$$\eta \hat{y}_i N_i = R_i L_i , \ \varsigma p F_a + \eta \hat{y}_a N_a = R_a L \quad \text{(A3)}$$

Adding the two equations in (A3), we obtain

$$\bar{r} = (\varsigma p F_a + \eta \hat{y}_i N_i + \eta \hat{y}_a N_a)/N. \quad \text{(A4)}$$

From $s_j = \bar{k}_j$ in (A1) and $s_j = \lambda \cdot \hat{y}_j$ we have $\hat{y}_j = \bar{k}_j / \lambda$. Substitute that and equation (A2) into equation (A4) we obtain

$$\bar{r} = (\alpha_0 K_a F_i / K_i + \eta K / \lambda_a)/N. \quad \text{(A5)}$$

where we used equation (13) and $\alpha_0 \equiv \alpha_i \varsigma / \alpha_a$. Substitute $p \cdot c_{ja} = \mu \cdot \hat{y}_j$ in (10) into equation (14)

$$\mu \cdot K = \lambda \cdot p \cdot F_a. \quad \text{(A6)}$$

Substituting equations (10) into equation (15) yields

$$\delta_0 \cdot K = F_i \quad \text{(A7)}$$

where $\delta_0 \equiv \xi / \lambda + \delta_k$. Substituting equations (A2) and (A7) into equation (A6) yields

$$K_a = \delta_\lambda \cdot K_i, \quad \text{(A8)}$$

where $\delta_\lambda \equiv \mu \cdot \alpha_a / \alpha_i (\xi + \delta_k \cdot \lambda)$. From equations (A8) and (13), we have

$$K = (1 + \delta_\lambda) K_i \quad \text{(A9)}$$

Equations (A8) and (A9) determine K_a and K as unique functions of K_i. Substitute equations (A7) into (A9) into (A5)

$$\bar{r} = \bar{r}_0 K_i , \quad \text{(A10)}$$

where

$$\bar{r}_0 = (1 + \delta_\lambda) \cdot (\alpha_0 \delta_\lambda \delta_0 + \eta / \lambda)/N.$$

From equations (4), (A7) and (A9), we have

$$r = \alpha_i \delta_0 (1 + \delta_\lambda) - \delta_k , \ w_i = \frac{\beta_i \delta_0 (1 + \delta_\lambda) K_i}{N_i}. \quad \text{(A11)}$$

From equations (2) and (A8), we obtain

$$pF_a = \left(\frac{r + \delta_k}{\alpha_a} \right) \delta_\lambda K_i , \ w_a = \beta_a \delta_\lambda \left(\frac{r + \delta_k}{\alpha_a} \right) \frac{K_i}{N_a}. \quad \text{(A12)}$$

Inserting equations (10) into utility functions (8) and then applying equation (12), we obtain

$$\left(\frac{R_i}{R_a} \right)^{\eta} = \frac{\bar{\theta}_i^{\rho} \bar{k}_i N_i^{\rho d_i}}{\bar{\theta}_a^{\rho} \bar{k}_a N_a^{\rho d_a}}, \quad \text{(A13)}$$

in which we use $\hat{y}_j = \bar{k}_j/\lambda$ and equations (9). Substitute $\hat{y}_j = \bar{k}_j/\lambda$ and $p \cdot F_a$ in (A12) into equations (A3)

$$R_i = m_1 \bar{k}_i N_i, \quad R_a = m_2 K_i + m_3 \bar{k}_a N_a, \tag{A14}$$

where

$$m_1 \equiv \frac{\eta}{\lambda L_i}, \quad m_2 \equiv \left(\frac{r + \delta_k}{\alpha_a}\right)\frac{\varsigma \delta_\lambda}{L}, \quad m_3 \equiv \frac{\eta}{\lambda L}.$$

Insert equations (A14) into equation (A13)

$$\left(\frac{m_1 \bar{k}_i N_i}{m_2 K_i + m_3 \bar{k}_a N_a}\right)^\eta = \frac{\bar{\theta}_i^\rho \bar{k}_i \, N_i^{\rho d_i}}{\bar{\theta}_a^\rho \bar{k}_a \, N_a^{\rho d_a}}. \tag{A15}$$

From $\hat{y}_j = \bar{k}_j/\lambda$ and equations (A10) and (6), we have $w_j + \bar{r}_0 K_i = r_1 \bar{k}_j$ where $r_1 \equiv 1/\lambda - r$. Substituting equations (A8), (A11) and (A12) into the above equations, we have

$$\bar{k}_i \equiv (m_5 + \bar{r}_0 N_i)\frac{K_i}{r_1 N_i}, \quad \bar{k}_a = (m_6 - \bar{r}_0 N_i)\frac{K_i}{r_1 (N - N_i)}, \tag{A16}$$

where

$$m_5 \equiv \beta_i \delta_0 (1 + \delta_\lambda), \quad m_6 \equiv (r + \delta_k)\frac{\beta_a \delta_\lambda}{\alpha_a} + \bar{r}_0 N.$$

Substitute equations (A16) yields into equation (A15)

$$\Phi_N(N_i) \equiv \left(\frac{m_1 (m_5 + \bar{r}_0 N_i)}{r_1 m_2 + m_3 (m_6 - \bar{r}_0 N_i)}\right)^\eta - \frac{\bar{\theta}_i^\rho N_i^{\rho d_i - 1}}{\bar{\theta}_a^\rho (N - N_i)^{\rho d_a - 1}}\left(\frac{m_5 + \bar{r}_0 N_i}{m_6 - \bar{r}_0 N_i}\right) = 0, \tag{A17}$$

where we use $N_a + N_i = N$ and equations (13) and (A9). This equation contains a single variable. The labour distribution is determined by a positive N_i such that

$$\Phi_N(N_i) = 0, \; 0 < N_i < N.$$

We require $\rho d_j - 1 < 0$, $j = a, i$. As $\Phi_N(N) > 0$ we see that the problem has at least one meaningful solution. As it is difficult to discuss conditions whether the problem has a unique solution, we will confirm whether the labour distribution is unique when simulating the model. From equation (A17) and $N_a = N - N_i$ we determine the labour distribution as a function of the population and other parameters.

For any given N_i from equations (3) and (4), we solve K_i as a function of H

$$K_i = m_0 H^{m_i/\beta_i}, \tag{A18}$$

in which

$$m_0 \equiv \left(\frac{\alpha_i A_i}{r + \delta_k}\right)^{1/\beta_i} N_i.$$

From equations (A17) and (A11) we may consider m_0 as a parameter. From equations (A16) we solve $\bar{k}_i$ and $\bar{k}_a$ as functions of H

$$\bar{k}_j = q_j H^{m_i/\beta_i}, \quad j = a, i, \tag{A19}$$

where

$$q_i \equiv (m_5 + \bar{r}_0 N_i)\frac{m_0}{r_1 N_i}, \quad q_a \equiv (m_6 - \bar{r}_0 N_i)\frac{m_0}{r_1 (N - N_i)}.$$

From equations (A14) and (A16), we get $R_a = m_4 K_i$, where $m_4 \equiv m_2 + (m_6 - \bar{r}_0 N_i) m_3 / r_1$. From $R_a = m_4 K_i$ and equations (2) and (A12), we obtain

$$L_a = \frac{\varsigma \delta_\lambda}{m_4} \frac{r + \delta_k}{\alpha_a}.$$

We now determine H. Substituting equations (1) and (3) into the last equation in (A1), we obtain the following equation

$$\Phi_h(H) \equiv \Phi_a(H) + \Phi_i(H) - \delta_h = 0, \tag{A20}$$

where we use equations (A8) and (A18) and

$$\Phi_a(H) \equiv \frac{\tau_a A_a N_a^{1+\beta_a} L_a^{\varsigma} \delta_\lambda^{\alpha_a} m_0^{\alpha_a}}{N} H^{x_a}, \quad \Phi_i(H) \equiv \frac{\tau_i A_i N_i^{1+\beta_i} m_0^{\alpha_i}}{N} H^{x_i},$$

$$x_a \equiv \frac{\alpha_a m_i}{\beta_i} + m_a - \varepsilon_a - 1, \quad x_i \equiv \frac{m_i}{\beta_i} - \varepsilon_i - 1.$$

We omit the case of $x_a = x_i = 0$. Equilibrium of the system is given by a positive H such that $\Phi(H) = 0$. When $x_a > 0$ and $x_a > 0$ equation $\Phi(H) = 0$ has a unique positive solution as $\Phi' = 0$ for any positive H, $\Phi(H) < 0$ and $\Phi(\infty) > 0$. Similarly, if $x_a < 0$ and $x_i < 0$ the equation $\Phi(H) = 0$ has a unique positive solution. It is easy to check that if either $x_a = 0$, or $x_i = 0$ then the system has a unique positive solution under certain conditions. We now prove that if $x_a > 0$and $x_i < 0$ (or $x_a < 0$ and $x_i > 0$), then the system has either two solutions or no solution. It is sufficient for us to examine one case, for instance that with $x_a > 0$ and $x_i < 0$. Since $\Phi(H) > 0$, $\Phi(\infty) > 0$ we see that $\Phi(H) = 0$ cannot have a unique solution. That is, the equation $\Phi(H) = 0$ has either multiple solutions, or no solution. On the other hand, as $\Phi'(H) = 0$ has a unique positive solution, we conclude that $\Phi(H) = 0$ has two solutions if $\Phi(H)$ has solutions. The necessary and sufficient condition for the existence of two solutions is that there exists a positive value H_1 of H such that $\Phi(H_1) < 0$ and $\Phi'(H_1) = 0$. We have thus proved the proposition.

A2: DESCRIBING THE MOTION WITH THREE DIFFERENTIAL EQUATIONS

We now show a procedure to determine dynamic properties of the system. We omit time index in expressions in Appendix A2. Similar to equation (A4), we have

$$\bar{r} = \frac{\alpha_0 K_a F_i / K_i + \eta \hat{y}_i N_i + \eta \hat{y}_a N_a}{N}, \tag{A21}$$

where we use equation (A2) and $\alpha_0 \equiv \alpha_i \varsigma / \alpha_a$. Substitute $p \cdot c_{ja} = \mu \cdot \hat{y}_j$ in (10) into equation (14)

$$\mu \left(\hat{y}_a N_a + \hat{y}_i N_i \right) = p F_a. \tag{A22}$$

Substituting equations (10) into equation (15) yields

$$\left(\xi + \lambda \right) \left(\hat{y}_a N_a + \hat{y}_i N_i \right) = F_i + \delta K. \tag{A23}$$

From equations (A22) and (A23), we have

$$\frac{F_i + \delta K}{p F_a} = \frac{\xi + \lambda}{\mu}. \tag{A24}$$

Substituting equation (A2) into equation (A24) yields

$$K_a = \frac{\alpha_a \mu}{\alpha_i (\xi + \lambda)} \frac{(F_i + \delta K) K_i}{F_i}. \tag{A25}$$

From equation (A25) and $K = K_a + K_i$ we solve

$$K = \frac{(1 + \alpha) K_i F_i}{F_i - \alpha \delta K_i}, \tag{A26}$$

in which $\alpha \equiv \alpha_a \mu / \alpha_i (\xi + \lambda)$. Equation (A26) determines $K(t)$ as a unique function of $K_i(t)$ and $H(t)$. We see that $K_a(t)$ is also determined as a function of $K_i(t)$ and $H(t)$. Substitute equation (A21) into the definitions of $\hat{y}_j$ in (6), we obtain

$$\hat{y}_j = (1 + r)\bar{k}_j + w_j + \frac{\alpha_0 K_a F_i / K_i + \eta \hat{y}_i N_i + \eta \hat{y}_a N_a}{N}, \quad j = a, i. \tag{A27}$$

Solving previous equations, we obtain

$$\hat{y}_i = \frac{(1 + r)N\bar{k}_i + N w_i + \alpha_0 K_a F_i / K_i + \eta N_a \bar{k}}{(1 - \eta)N}, \quad \hat{y}_a = \bar{k} + \hat{y}_i, \tag{A28}$$

in which $\bar{k} = (1 + r)(\bar{k}_a - \bar{k}_i) + w_a - w_i$. From equations (4), we see that r and w_i can be considered as functions of K_i, H and N_i. From equations (A3), we have

$$w_a = \frac{\beta_a (r + \delta_k) K_a}{\alpha_a N_a}. \tag{A29}$$

By equations (A2) and (A29), p and w_a are also functions of K_i, H and N_i. Using $N_a = N - N_i$ and equations (A2) and (A29), we can express $\hat{y}_i$ as a unique function of $\bar{k}_a$, $\bar{k}_i$, K_i, H and N_i:

$$\bar{k} = \Psi_k (\bar{k}_i, \bar{k}_a, K_i, H, N_i),$$
$$\hat{y}_i = \Psi (\bar{k}_i, \bar{k}_a, K_i, H, N_i). \tag{A30}$$

Inserting equations (10) into utility functions (8) and then applying equation (12), we obtain

$$\left(\frac{R_i}{R_a}\right)^{\eta} = \frac{\bar{\theta}_i \hat{y}_i N_i^{d_i}}{\bar{\theta}_a \hat{y}_a N_a^{d_a}}, \tag{A31}$$

in which we also use (9). Substitute equations (A3) into equation (A31)

$$\left(\frac{\alpha_0 K_a F_i}{K_i} + \eta \hat{y}_a N_a\right)^{-\eta} = \bar{\theta} \frac{\hat{y}_i^{1-\eta} N_i^{d_i - \eta}}{\hat{y}_a N_a^{d_a}}, \tag{A32}$$

where we use (A2) and

$$\bar{\theta} \equiv \frac{\bar{\theta}_i}{\bar{\theta}_a} \left(\frac{L_i}{\eta L_a}\right)^{\eta}.$$

Substitute equations (A28) into equation (A30)

$$\bar{\Psi}(\bar{k}_i, \bar{k}_a, K_i, H, N_i) \equiv \left(\frac{\alpha_0 K_a F_i}{K_i} + \eta (N - N_i)(\Psi_k + \Psi)\right)^{-\eta} - \frac{\bar{\theta} \Psi^{1-\eta} N_i^{d_i - \eta}}{(\Psi_k + \Psi)(N - N_i)^{d_a}} = 0. \tag{A33}$$

Assume that from equation (A33) we determine N_i as a function of $\bar{k}_a$, $\bar{k}_i$, K_i and H:

$$N_i \equiv \bar{\Psi}_N (\bar{k}_i, \bar{k}_a, K_i, H). \tag{A34}$$

From equation (A34) and $N_a = N - N_i$, we determine the labour distribution as functions of $\bar{k}_a$, $\bar{k}_i$, K_i and H. From equation (13) and $N_a = N - N_i$ we have

$$K_a + K_i = \bar{k}_a N + (\bar{k}_i - \bar{k}_a) \bar{\Psi}_N (\bar{k}_i, \bar{k}_a, K_i, H). \tag{A35}$$

Equation (A35) contains four variables: $\bar{k}_a$, $\bar{k}_i$, K_i and H. Assume that we solve $\bar{k}_a$ as a function of $\bar{k}_i$, K_i and H as follows

$$\bar{k}_a = \bar{\Psi}_0 (\bar{k}_i, K_i, H). \tag{A36}$$

By the following procedure, we can determine all the variables as functions of $\bar{k}_i(t)$, $K_i(t)$ and $H(t)$ at any point of time: $\bar{k}_a$ by equation (A36) → N_i by equation (A34) → $N_a = N - N_i$ → $\bar{k}$ and $\hat{y}_i$ by equations (A30) → r and w_i by equations (4) → K by (A26) → p by equation (A2) → $\bar{r}$

by equation (A27) → w_a by (A29) → $\hat{y}_a$ by (A0) → $l_i = L_i/N_i$ → R_a by (A3) → l_a and R_i by equations (10) → c_a, c_{aa} and s_a by equations (10) → c_i, c_{ia} and s_i by equations (10) → F_i by equation (3) → F_a by equation (1). From equations (11), (17) and (A36), we have

$$\dot{\bar{k}}_a(t) = \Lambda_a\left(\bar{k}_i(t),\, K_i(t),\, H(t)\right) \equiv s_a(t) - \overline{\Psi}_0, \tag{A37}$$

$$\dot{\bar{k}}_i(t) = \Lambda_i\left(\bar{k}_i(t),\, K_i(t),\, H(t)\right) \equiv s_i(t) - \bar{k}_i(t),$$

$$\dot{H}(t) = \Lambda_H\left(\bar{k}_i(t),\, K_i(t),\, H(t)\right) \equiv \frac{\tau_a N_a(t) F_a(t)}{N H^{\varepsilon_a}(t)} + \frac{\tau_i N_i(t) F_i(t)}{N H^{\varepsilon_i}(t)} - \delta_h H(t). \tag{A38}$$

Taking derivatives of equation (13) with respect to t we obtain

$$\dot{\bar{k}}_a = \overline{\Psi}_{0k}\dot{\bar{k}}_i + \overline{\Psi}_{0K}\dot{K}_i + \overline{\Psi}_{0H}\dot{H}, \tag{A39}$$

where $\overline{\Psi}_{0k}$, $\overline{\Psi}_{0K}$ and $\overline{\Psi}_{0H}$ are partial derivatives of $\overline{\Psi}_0$ with respect to $\bar{k}_i(t)$ $K_i(t)$ and $H(t)$, respectively. From equations (A37) and (A39), we delete $\dot{\bar{k}}_a$ and obtain

$$\dot{K}_i(t) = \Lambda_K\left(\bar{k}_i(t),\, K_i(t),\, H(t)\right) \equiv \frac{\Lambda_a - \overline{\Psi}_{0H}\Lambda_H - \overline{\Psi}_{0k}\Lambda_i}{\overline{\Psi}_{0K}}. \tag{A40}$$

Equations (A38) and (A40) contain three variables: $\bar{k}_i(t)$, $K_i(t)$ and $H(t)$. The three differential equations determine the motion of $\bar{k}_i(t)$, $K_i(t)$ and $H(t)$ over time. All other variables are determined as functions of the three variables at any point of time. As the expressions are tedious, t is difficult to interpret analytical results. We are concerned only with equilibrium issues.

ACKNOWLEDGMENT

The author is grateful to important comments of two anonymous referees.

REMARKS

[1]The model was first presented in [9] and [10]. The original static model has been extended in different ways (see for instance, [10 – 12]). As mentioned by Fields [13], the model has been extended to allow for an urban informal sector, on-the-job search from agriculture, duality within the rural sector, educational differences among workers, job fixity, mobile capital, endogenous urban wage setting, risk-aversion, a system of demand of goods and many other factors. The list of extensions can be much longer.

[2]The assumption by Matsuyama is not supported by the empirical evidence presented in [14] and [15]. It is demonstrated that growth in total factor productivity in agriculture is not only strictly positive but, in most cases, larger than total factor productivity growth in industry. It should also be remarked that the two-sector model presented in [8] fixes the saving rate and does not consider endogenous change in human capital.

[3]The assumption of full utilization of factor resources is strict. However, as shown in [3] for a two-sector economy with constant human capital, it is conceptually not difficult to relax the assumption of full employment of labour force. Nevertheless, the model with unemployment and human capital will become difficult to analyze.

[4]Although this assumption is often accepted in the literature of urbanization with agriculture (see [17, 18]), some studies try to examine impact of transportation costs upon urban-rural labour distribution (e.g., [19, 20]).

[5]As the urban land used for industrial sector is not large, the omission of industrial land use is acceptable.

[6]Zhang has also examined the relations between his approach and the Solow growth theory, the Ramsey growth theory, the permanent income hypothesis, and the Keynesian consumption function in details.

[7]The concept of amenity is often used in the literature of urban and regional economics (see, for instance, [6, 21 – 24]). The concept has recently been introduced into the Ramsey growth model in [25].

[8]The Keynesian consumption function and permanent income hypotheses (which are not the same) are similar to our approach in the sense that the propensity to save is affected by wealth. It should be noted that Zhang's approach is very general in the sense that by introducing endogenous taste change, Zhang's approach generates the same consumer behaviour as described by the traditional approaches (see [3]).

[9]Another important issue is about taste change. In any basic course in microeconomics, concepts of normal, inferior, and luxury goods are introduced. For illustration, we now point out possible ways to take account of a household's preference change due to changes in income. Let there be n kinds of goods and services. The household's utility function is given, for instance, by

$$U(t) = s^{\lambda(t)}(t)\prod_{j=1}^{n} c_j^{\xi_j(t)}(t),$$

where $c_j(t)$ is the consumption level of goods j, $s(t)$ is the saving, and the preference parameters are defined similarly as in (8). The budget constraint is given by

$$\sum_{j=1}^{n} p_j(t)c_j(t) + s(t) = \hat{y}(t)$$

where $\hat{y}(t)$ is the disposable income. The optimal solution is

$$s(t) = \lambda(t)\hat{y}(t),\ c_j(t) = \xi_j(t)\,\hat{y}(t)/p_j(t),\quad j = 1, \ldots, n.$$

Here, we consider that the propensities are influenced by the household's disposable income (and/or wage and wealth), his age, and other factors like relative social status in the following way:

$$\lambda(t) = \lambda[\hat{y}(t), t],\ \xi_j(t) = \xi_j[\hat{y}(t), t],\quad \mathrm{j} = 1, \ldots, n.$$

For instance, if good 1 is an inferior good, and the others are normal, we may specify the preference change as follows: $\xi_1(t) = \xi_{10} - \xi_{11}\cdot\hat{y}(t)$, $\xi_1(t) > 0$, where ξ_{10} and ξ_{11} are constants and the rest of the parameters are kept constant. The preference change may be nonlinear. We will not examine taste change in this study as the analysis is already very complicated.

[10]In the contemporary literature of growth theory, different sources of human capital, such as education, are introduced to explain economic growth and development (see, e.g. [26 – 29]). This study is limited the case of learning by doing. It should be noted that Zhang [30] takes account of three sources of learning, learning by doing, learning by leisure, and learning by education.

[11]For simplicity, we assume a linear relation between the outputs and growth rate of human capital. It is important to examine what will happen to the system if the growth rate is related to the outputs with some reasonable nonlinear relations.

[12]Although we failed to explicitly give stability conditions, Appendix A2 shows the procedure of finding out the dynamic equations of the economic system.

[13]As mentioned before, the main extension of this study is to introduce amenity differences between the rural and urban areas (which are the key factors for explaining wage, consumption and land rent differences). In [3] the total land is not fixed and the transformation form one type of land use to another is costless and instantaneous.

[14]This assumption is accepted, for instance in [5].

[15]The specification is strict. For instance, as the urban area is expanded, the city may become more attractive.

[16]These properties are mainly due to the specified forms of the utility and production functions.

[17]The problem of increasing demand for food coupled with diminishing returns in agriculture was central to the classical growth theories of Malthus and Ricardo. In [10], Panagariya and Succar introduce economies of scale to the Harris-Todaro framework with fixed capital within a static framework.

[18]See [3] for more detailed discussions on multiple equilibrium points with different levels of human capital.

[19]We also demonstrate that the urban amenity parameter is improved, some people will migrate from the rural area to the urban area. The urban lot size falls and the rural lot size and agricultural land use are increased. The effects of the urban amenity improvement are similar to those caused by the productivity improvement.

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MODEL RASTA DVA SEKTORA S ENDOGENIM LJUDSKIM RESURSIMA I MOGUĆNOSTIMA

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SAŽETAK

Rad razmatra pitanja vezana uz urbanizaciju s migracijom radne snage. Glavna odstupanja od tradicionalnih pristupa dinamici ekonomskih struktura su što se u radu koristi alternativni pristup ponašanju potrošača te što se uvodi akumulacija ljudskog kapitala putem učenja stečenog djelovanjem. Model opisuje dinamičko međudjelovanje između poljoprivredne i industrijske proizvodnje, ruralne i urbane mogućnosti, distribuciju faktora proizvodnje i preferencija kao i akumulaciju endogenog kapitala i ljudskih resursa. Pokazujemo kako dinamički sustav može imati ili jedno, ili više ravnotežnih stanja, ovisno o povratku na skalu u dva sektora. Također smo ispitali učinke promjena pojedinih parametara modela.

KLJUČNE RIJEČI

Model dva sektora, poljoprivredni sektor, industrijski sektor, fizička akumulacija kapitala, akumulacija ljudskih resursa, ruralne i urbane mogućnosti

Interdisciplinary Description of Complex Systems 6(2), 117-131, 2008

RESPONSE AND DYNAMICAL STABILITY OF OSCILLATORS WITH DISCONTINUOUS OR STEEP FIRST DERIVATIVE OF RESTORING CHARACTERISTIC

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Regular article *Received:* 2. August 2008. *Accepted:* 8. January 2009.

ABSTRACT

Response and dynamical stability of oscillators with discontinuous or steep first derivative of restoring characteristic is considered in this paper. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic force excitation is analysed by the method of piecing the exact solutions (MPES) in the time domain and by the incremental harmonic balance method (IHBM) in the frequency domain. The stability of the periodic solutions obtained in the frequency domain by IHBM is estimated by the Floquet-Lyapunov theorem. Obtained frequency response characteristic is very complex and includes multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued and non-periodic solutions. Determining of frequency response characteristic in the time domain by MPES is exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions. In the frequency domain, IHBM is very efficient and very well suited for obtaining wide range frequency response characteristics, parametric studies and bifurcation analysis. On the other hand, neglecting of very small harmonic terms (which in-significantly influence the r.m.s. values of the response and are very small in comparison to other terms of the spectrum) can cause very large error in evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution. Moreover, frequency ranges are detected inside which the procedure of evaluation of eigenvalues of the monodromy matrix does not converge with increasing the number of harmonics included in the supposed approximate solution.

KEY WORDS

dynamical stability, response characteristic, non-linear vibrations, piecewise-linear system

CLASSIFICATION

ACM: J.2 Engineering

JEL: Z0

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INTRODUCTION

Among the great number of various types of non-linear dynamic systems a very specific group constitutes non-linear systems described by differential equations which contain non-linear restoring characteristic with discontinuous or steep first derivative (for example systems with clearances, rolling bearings, gears, clutches, impacting oscillators, etc.). Frequency response characteristics of these systems are usually very complex and include multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued solutions, and possibility of non-periodic solutions. Both periodic and non-periodic responses can be determined in the time domain by using digital simulation. But procedures of that kind can be exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions (where many combinations of initial conditions have to be examined for obtaining all possible steady-state solutions), for lightly damped systems (since a great number of excitation periods must be simulated to obtain a steady-state response), and when the state of the system is near to bifurcation. These methods are not suitable for obtaining wide range frequency response characteristics, unstable solutions and for bifurcation analysis also. A very efficient method for solving strong non-linear differential equations in the frequency domain is the harmonic balance method (HBM) [1-6]. When the assumption of dominance of primary resonance in the response is satisfied, the HBM (single harmonic) is very accurate and numerically very efficient method for obtaining periodic response of non-linear systems with harmonic excitation. But it becomes very inaccurate if the influence of higher harmonics in the response is significant. Multi frequency harmonic balance methods (e.g., Incremental harmonic balance method or Newton-Raphson harmonic balance method [7-14]) provide the study of effects of superharmonics and subharmonics to response. These methods become exceptionally efficient in combination with path following techniques [15-17], and can be successfully applied to a wide range of non-linear problems. They are very well suited for parametric studies because a new solution can be sought by these methods, with the previous solution used as a very good approximation. Since these methods enable obtaining both dynamically stable and unstable solutions, determining of dynamical stability of these solutions should be reliable and numerically efficient. Since the estimation of dynamical stability of the steady state response by Floquet-Lyapunov theorem [18] is a sensitive procedure [19-21], the factors which can lead to incorrect prediction of the dynamical stability must be taken into consideration.

Responses determined in the time domain (MPES) and in the frequency domain (HBM and IHBM) are considered in this paper as well as problems which can occur in estimation of dynamical stability of the periodic solutions obtained in the frequency domain. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic excitation is analysed.

MODEL OF A MECHANICAL SYSTEM WITH A CLEARANCE

Model of a simple mechanical system with clearance is shown in Figure 1. It consists of an inertia element m, a linear viscous damping parameter c, and a non-linear elastic element defined by a piecewise-linear function $g(x)$ and a coefficient k. When the system is excited by a periodic harmonic force $F(t)$, the motion of the system can be described by the non-linear differential equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+c\frac{\mathrm{d}x}{\mathrm{d}t}+kg\left(x\right)=F(t)=F_m+F_p\cos\left(\Omega t+\varphi_F\right)=f_0+f_C\cos\left(\Omega t\right)+f_S\sin\left(\Omega t\right), \quad (1)$$

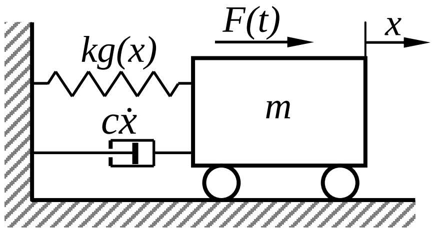


Figure 1. Model of vibration system.

where f_0 = F_m represents mean transmitted force, $F_p = \sqrt{f_C^2 + f_S^2}$ is the amplitude of the vibratory component at frequency Ω, while f_C and f_S are force component amplitudes of the corresponding harmonic terms and φ_F is the excitation phase angle.

The piecewise linear function $g(x)$ and its derivative are shown in Figure 2(a) and Figure 2(b), respectivelly. Parameter b denotes one-half of the clearance space. Since the procedure of prediction of the dynamical stability is based on derivative of a non-linear function, expressions for non-linear function and its derivative are given:

$$g(x) = h^*(x - b^*), \tag{2}$$

$$\frac{\partial g(x)}{\partial x} = h^*, \tag{3}$$

where

$$h^* = \begin{Bmatrix} 1, & b < x \\ 0, & -b \le x \le b \\ 1, & x < -b \end{Bmatrix}, \quad b^* = \begin{Bmatrix} b, & b < x \\ 0, & -b \le x \le b \\ -b, & x < -b \end{Bmatrix}. \tag{4}$$

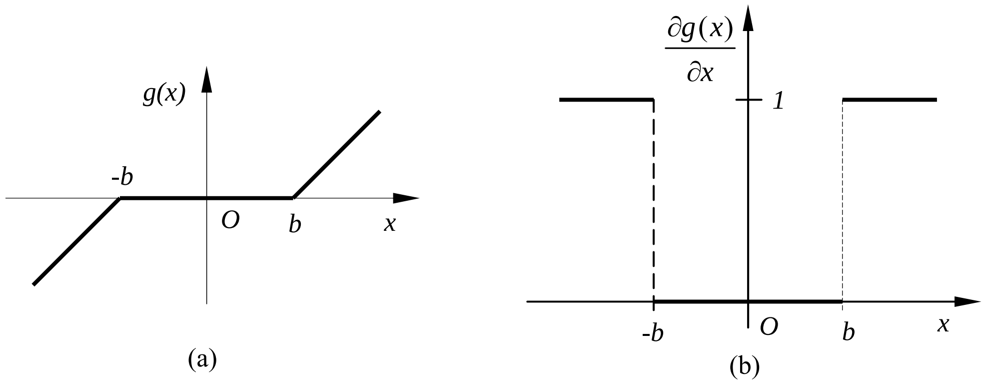


Figure 2. Graphs showing a) non-linear function $g(x)$ and b) its derivative $\partial g(x)/\partial x$.

BRIEF DESCRIPTION OF THE APPLIED METHODS

THE INCREMENTAL HARMONIC BALANCE METHOD (IHBM)

By introducing a non-dimensional time θ as a new independent variable, the differential equation (1) can be rewritten in the non-dimensional form:

$$\frac{\eta^2}{\nu^2}\frac{d^2\bar{x}}{d\theta^2} + \frac{2\zeta\eta}{\nu}\frac{d\bar{x}}{d\theta} + g(\bar{x}) = \bar{F}(\theta) = \sum_{n=0}^{M}(\bar{f}_n \cos n(\nu\theta) + \bar{g}_n \sin n(\nu\theta)), \tag{5}$$

with

$$\bar{x} = \frac{x}{l},\ \bar{b} = \frac{b}{l},\ \omega_0 = \sqrt{\frac{k}{m}},\ \zeta = \frac{c}{2m\omega_0},\quad \bar{f}_C = \frac{f_C}{ml\omega_0^2},\ \bar{f}_S = \frac{f_S}{ml\omega_0^2},$$

$$\eta = \frac{\Omega}{\omega_0},\ \tau = \omega_0 \cdot t,\ \Omega t = \eta\tau = \nu\theta\ .$$

In this way, the period of the response (with ν subharmonics taken in consideration) is always 2π, making it possible (by using the IHBM) to consider any number of superharmonics and subharmonics included in the supposed approximate solution. Any characteristic dimension of the system is denoted by l here.

Supposed approximate solution is given by:

$$\bar{x} = \sum_{\mathrm{i}=0}^{\mathrm{N}} a_{\mathrm{i}} \cos \mathrm{i}\theta + b_{\mathrm{i}} \sin \mathrm{i}\theta = \boldsymbol{T}\,\boldsymbol{a},, \tag{6}$$

where

$$\boldsymbol{T} = \left[1,\ \cos\theta,\ \cos 2\theta, ..., \cos \mathrm{N}\theta,\ \sin\theta,\ \sin 2\theta, ..., \sin \mathrm{N}\theta\right],$$

$$\boldsymbol{a} = \left[a_0, a_1, ..., a_N, b_1, b_2, ..., b_N\right]^T .$$

The equation $N = \nu M$ represents the number of all harmonics included in the supposed solution, ν is the number of subharmonics and M is the number of superharmonics. By applying this method, which consists of two basic steps: incrementation and Galerkin's procedure, the non-linear differential equation (5) is transformed into the system of $2N + 1$ linearized incremental algebraic equations:

$$\mathsf{K}^{\mathrm{j}}\, \Delta\boldsymbol{a}^{\mathrm{j}+1} = \boldsymbol{r}^{\mathrm{j}}, \tag{7}$$

$$\boldsymbol{a}^{\mathrm{j}+1} = \boldsymbol{a}^{\mathrm{j}} + \Delta\boldsymbol{a}^{\mathrm{j}+1}, \tag{8}$$

with Fourier coefficients (a_0, a_{i}, b_{i}, i = 1, …, N) as unknowns. In equations (7) and (8), the superscript j denotes the number of iterations. In each incremental step, only linear (i.e., linearized) algebraic equations have to be formed and solved. A solution is obtained from the iteration process when the corrective vector norm $\|r\|$ is smaller than a certain (arbitrary) convergence criterion. The comprehensive description of the method, its application to piecewise-linear systems and the way of determining elements of Jacobian matrix K and the corrector *r* in explicit form is given by Wong et al. in [10]. Generally, accuracy of the approximate solution obtained by using IHBM depends on the number of harmonics included in the solution, accuracy of procedures used for determining elements of K and *r*, and a value of convergence criterion. Since the IHBM described by Wong et al. ([10]) is used in this work, accuracy of the procedure of determining elements of K and *r* depends only on the precision of numerical determination of times θ_{i} in which the system changes stage stiffness region (Fig. 3). Regarding the parameter *b*, the three stages in the problem are defined in accordance with (4).

THE METHOD OF PIECING THE EXACT SOLUTIONS (MPES)

By introducing the non-dimensional time τ as an independent variable (what is convenient when one determines response in the time domain), the differential equation (1) can be rewritten in the non-dimensional form:

$$\frac{\mathrm{d}^2\,\bar{x}}{\mathrm{d}\,\tau^2} + 2\zeta\frac{\mathrm{d}\,\bar{x}}{\mathrm{d}\,\tau} + g(\bar{x}) = \bar{F}_m + \bar{F}_p \cos(\eta\,\tau + \varphi_F), \tag{9}$$

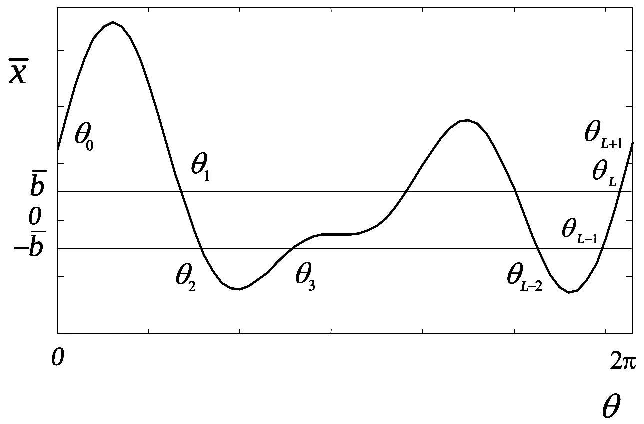


Figure 3. Solutions to the equations $\overline{x} = \overline{b}$ and $\overline{x} = -\overline{b}$.

where

$$\overline{F}_m = \frac{F_m}{ml\omega_0^2}, \quad \overline{F}_p = \frac{F_p}{ml\omega_0^2}.$$

Force-displacement relationship $g(\overline{x})$, shown in Fig. 2(a), is piecewise-linear. Local solutions of differential equation (9) are known explicitly inside each of the stage stiffness, and can be repeatedly matched at $\overline{x} = \overline{b}$ and $\overline{x} = -\overline{b}$, to obtain a global solution of (9). Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in a closed form. But, the matching of local solutions can be numerically done very easily. Only approximation done by the applying of this procedure is in the numerical determination of times in which the system changes stage stiffness region ($\overline{x} = \overline{b}$, $\overline{x} = -\overline{b}$). Effective amplitudes $\overline{x}_p$ of the steady state time domain responses are calculated by using:

$$\overline{x}_m = \frac{1}{T}\int_{-T/2}^{T/2} \overline{x}(\tau)\,\mathrm{d}\tau\,, \quad \overline{x}_p = \sqrt{2}\sqrt{\frac{1}{T}\int_{-T/2}^{T/2}\left(\overline{x}(\tau) - \overline{x}_m\right)^2\,\mathrm{d}\tau}\,, \tag{10}$$

where T denotes the period of the response. The solutions of equation (9) inside each of the stage stiffness region are:

1. for $-\overline{b} \le \overline{x} \le \overline{b}$

$$\overline{x} = -\frac{C_1}{2\zeta}\mathrm{e}^{-2\zeta(\tau-\tau_0)} + \frac{\overline{F}_m}{2\zeta}\left(\tau - \tau_0\right) + \frac{Q_{1a}}{\eta}\sin\left(\eta\tau + \varphi_F - \varphi_R\right) + C_2\,, \tag{11a}$$

$$\frac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau} = C_1\mathrm{e}^{-2\zeta(\tau-\tau_0)} + \frac{\overline{F}_m}{2\zeta} + Q_{1a}\cos\left(\eta\tau + \varphi_F - \varphi_R\right), \tag{11b}$$

where

$$Q_{1a} = \frac{\overline{F}_p}{\sqrt{\eta^2 + 4\zeta^2}}, \quad \varphi_R = \tan^{-1}\left(\frac{\eta}{2\zeta}\right),$$

$$C_1 = \left(\frac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 - \frac{\overline{F}_m}{2\zeta} - Q_{1a}\cos\left(\eta\tau_0 + \varphi_F - \varphi_R\right),$$

$$C_2 = \overline{x}_0 + \frac{C_1}{2\zeta} - \frac{Q_{1a}}{\eta}\sin\left(\eta\tau_0 + \varphi_F - \varphi_R\right).$$

in which $\bar{x}_0$ and $(\mathrm{d}\bar{x}/\mathrm{d}\tau)_0$ are non-dimensional displacement and velocity at the initial time $\tau = \tau_0$, i.e. at time in which the motion is determined by the given equation.

2. for $\bar{x} \geq \bar{b}$

$$\bar{x} = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[A_2\cos\sqrt{1-\zeta^2}(\tau-\tau_0) + B_2\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] + \bar{F}_m + \\ +(1-\alpha)\bar{b} + Q_2\cos(\eta\tau + \varphi_F - \varphi_R), \tag{12a}$$

$$\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right) = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[\left(B_2\sqrt{1-\zeta^2} - \zeta A_2\right)\cos\sqrt{1-\zeta^2}(\tau-\tau_0) - \left(A_2\sqrt{1-\zeta^2} + \zeta B_2\right)\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] - \\ -\eta Q_2\sin(\eta\tau + \varphi_F - \varphi_R), \tag{12b}$$

where

$$Q_2 = \frac{\bar{F}_p}{\sqrt{\left(1-\eta^2\right)^2 + (2\zeta\eta)^2}}, \quad \varphi_R = \tan^{-1}\left(\frac{2\zeta\eta}{1-\eta^2}\right),$$

$$A_2 = \bar{x}_0 - \bar{F}_m - (1-\alpha)\bar{b} - Q_2\cos(\eta\tau_0 + \varphi_F - \varphi_R),$$

$$B_2 = \frac{1}{\sqrt{1-\zeta^2}}\left[\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 + \zeta A_2 + \eta Q_2\sin(\eta\tau_0 + \varphi_F - \varphi_R)\right].$$

3. for $\bar{x} \leq -\bar{b}$

$$\bar{x} = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[A_3\cos\sqrt{1-\zeta^2}(\tau-\tau_0) + B_3\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] + \bar{F}_m - \\ -(1-\alpha)\bar{b} + Q_3\cos(\eta\tau + \varphi_F - \varphi_R), \tag{13a}$$

$$\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right) = \mathrm{e}^{-\zeta(\tau-\tau_0)}\left[\left(B_3\sqrt{1-\zeta^2} - \zeta A_3\right)\cos\sqrt{1-\zeta^2}(\tau-\tau_0) - \left(A_3\sqrt{1-\zeta^2} + \zeta B_3\right)\sin\sqrt{1-\zeta^2}(\tau-\tau_0)\right] - \\ -\eta Q_3\sin(\eta\tau + \varphi_F - \varphi_R), \tag{13b}$$

where

$$Q_{3j} = Q_{2j} = \frac{\bar{F}_p}{\sqrt{\left(1-\eta^2\right)^2 + (2\zeta\eta)^2}}, \quad \varphi_R = \tan^{-1}\left(\frac{2\zeta\eta}{1-\eta^2}\right),$$

$$A_3 = \bar{x}_0 - \bar{F}_m + (1-\alpha)\bar{b} - Q_3\cos(\eta\tau_0 + \varphi_F - \varphi_R),$$

$$B_3 = \frac{1}{\sqrt{1-\zeta^2}}\left[\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 + \zeta A_3 + \eta Q_3\sin(\eta\tau_0 + \varphi_F - \varphi_R)\right].$$

THE STABILITY OF THE STEADY STATE SOLUTION

When the periodic solution is obtained, the stability of the given solution can be determined by examining the perturbed solution $\bar{x}^*$:

$$\bar{x}^* = \bar{x} + \Delta\bar{x}^*, \tag{14}$$

where $\Delta\bar{x}^*$ is a small perturbation of a periodic solution $\bar{x}$. By substitution of equation (14) into equation (5), and after expanding the non-linear function $g(\bar{x})$ in Taylor's series about the periodic solution with neglecting non-linear incremental terms, one obtains a linear homogeneous differential equation with time changing periodic coefficients $\partial g(\bar{x})/\partial\bar{x}$:

$$\frac{\eta^2}{\nu^2}\frac{\mathrm{d}^2\Delta\bar{x}^*}{\mathrm{d}\theta^2}+\frac{2\zeta\eta}{\nu}\frac{\mathrm{d}\Delta\bar{x}^*}{\mathrm{d}\theta}+\frac{\partial g(\bar{x})}{\partial\bar{x}}\Delta\bar{x}^*=0. \tag{15}$$

When the steady state solution $\bar{x}(\theta)$ is determined, the values of $\partial g(\bar{x})/\partial\bar{x}$ are known inside the period of the response. A very efficient and very often used method for determining the stability of the periodic solution is based on the Floquet-Lyapunov theorem [18, 22]. For that purpose equation (15) can be rewritten in the state variable form as:

$$\frac{\mathrm{d}\bar{\boldsymbol{X}}^*}{\mathrm{d}\theta}=\mathsf{A}(\theta)\,\bar{\boldsymbol{X}}^*, \tag{16}$$

where

$$\bar{\boldsymbol{X}}^*=\begin{Bmatrix}\Delta\bar{\boldsymbol{x}}^*\\ \dfrac{\mathrm{d}\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta}\end{Bmatrix},\quad \frac{\mathrm{d}\bar{\boldsymbol{X}}^*}{\mathrm{d}\theta}=\begin{Bmatrix}\dfrac{\mathrm{d}\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta}\\ \dfrac{\mathrm{d}^2\Delta\bar{\boldsymbol{x}}^*}{\mathrm{d}\theta^2}\end{Bmatrix},\quad \mathsf{A}(\theta)=\begin{bmatrix}0 & 1\\ -\dfrac{\nu^2}{\eta^2}\left(\dfrac{\partial\boldsymbol{g}(\bar{\boldsymbol{x}})}{\partial\bar{\boldsymbol{x}}}\right) & -\dfrac{2\nu\zeta}{\eta}\end{bmatrix}. \tag{17}$$

Since the matrix $\mathsf{A}(\theta)$ is a periodic function of θ with period 2π, the stability criteria are related to the eigenvalues of the monodromy matrix, which is defined as the state transition matrix at the end of one period. According to Floquet-Lyapunov theorem, the solution is stable if all the moduli of the eigenvalues of the monodromy matrix are less than unity. Otherwise the solution is unstable. Bifurcation occurs when one of the moduli of the eigenvalues of the monodromy matrix reaches unity. Generally, it is not possible to derive an analytic expression for the transition matrix. But, if the non-linear force-displacement relationship is piecewise-linear, its derivative $\partial g(x)/\partial x = h^*$ is, according to (4), constant inside each of the intervals $[\theta_i, \theta_{i+1}]$. Figure 3 shows a period of the response where $\theta_0 = 0$ and $\theta_{L+1} = 2\pi$. There are L times denoted as $\theta_1, \theta_2, \ldots, \theta_L$, in which the system undergoes a stiffness change. Consequently, $\mathsf{A}(\theta_i, \theta_{i+1})$ is also a constant matrix inside that interval. According to [23], for the constant $\mathsf{A}(\theta_i, \theta_{i+1})$ (inside the interval $[\theta_i, \theta_{i+1}]$), transition matrix $\Phi(\theta_{i+1}, \theta_i)$ can be expressed as:

$$\Phi(\theta_{i+1},\theta_i)=\mathrm{e}^{\mathsf{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}; \tag{18}$$

and for the whole interval $[0, 2\pi]$ according to [10] one obtains:

$$\left[\Phi(2\pi,0)\right]=\prod_{i=0}^{L}\mathrm{e}^{\mathsf{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}. \tag{19}$$

Beside the precision of numerical determination of times θi in which the system changes stage stiffness region ($\bar{x}=\bar{b}$, $\bar{x}=-\bar{b}$), the only approximation occurring in this procedure is the accuracy of computation of the matrix exponential $\exp[\mathsf{A}(\theta_i, \theta_{i+1})\cdot(\theta_{i+1}-\theta_i)]$ and the product of matrix exponentials $\prod_{i=0}^{L}\mathrm{e}^{\mathsf{A}(\theta_i,\theta_{i+1})\cdot(\theta_{i+1}-\theta_i)}$.

NUMERICAL EXAMPLES

Figures 4 and 5 show effective amplitude-frequency plots $\bar{x}_p=\bar{x}_p(\eta)$ obtained by MPES (both periodic and non-periodic solutions) for the parameter values: $\bar{b}=1$, $\zeta = 0{,}03$,

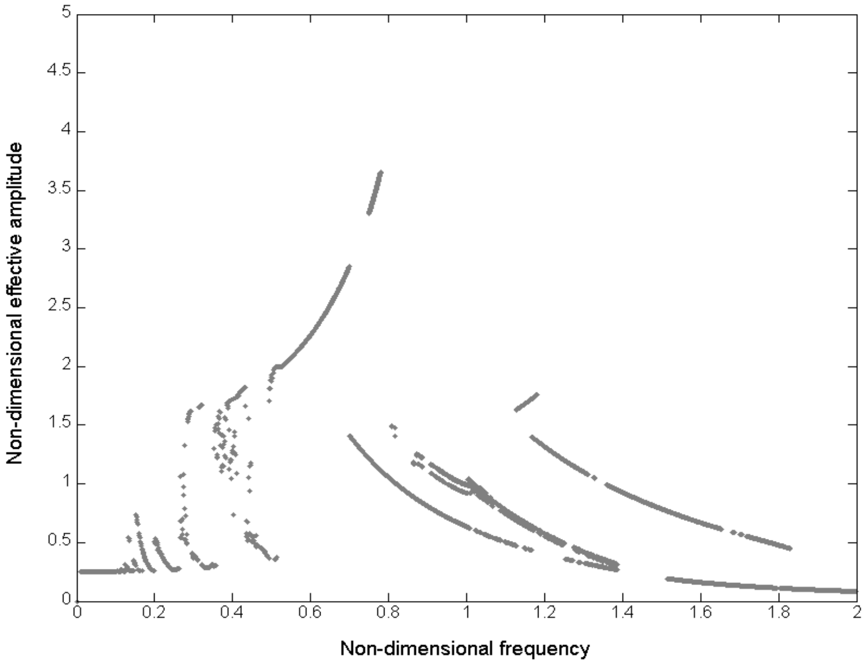


Figure 4. Effective amplitude-frequency plot $\overline{x}_p = \overline{x}_p\left(\eta\right)$ obtained by MPES: $\overline{x}_0 = 0$, $\left(\dfrac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 = 0$.

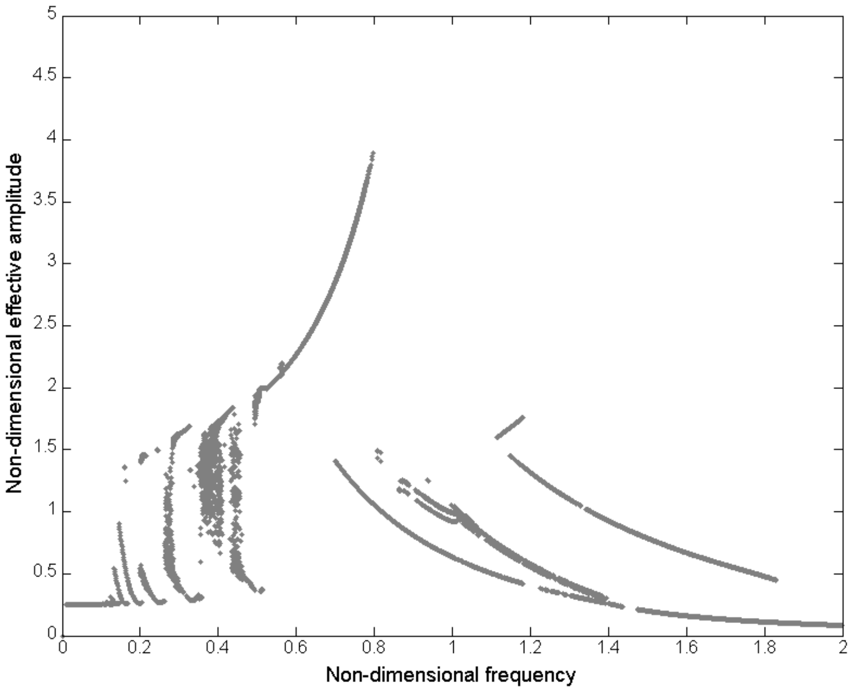


Figure 5. Effective amplitude-frequency plot $\overline{x}_p = \overline{x}_p\left(\eta\right)$ obtained by MPES for $\overline{x}_0 = 0$, $\left(\dfrac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 = 0$; $\overline{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 = 1$; $\overline{x}_0 = 1$, $\left(\dfrac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 = -1$ and $\overline{x}_0 = -1$, $\left(\dfrac{\mathrm{d}\,\overline{x}}{\mathrm{d}\,\tau}\right)_0 = -1$.

$\overline{f}_0 = \overline{F}_m = 0.25,\quad \overline{f}_C = \overline{F}_p = 0.25,\quad \overline{f}_S = 0 \quad \left(\varphi_F = 0\right)$. Figure 4 shows 1990 effective amplitudes $\overline{x}_p$ of the time domain responses obtained at 1990 non-dimensional frequencies η for initial conditions: $\bar{x}_0 = 0$ and $(\mathrm{d}\bar{x}/\mathrm{d}\tau)_0 = 0$. Figure 5 shows 7960 effective amplitudes $\bar{x}_p$ obtained at the same 1990 non-dimensional frequencies η, for four different initial conditions: $\bar{x}_0 = 0$, $\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 0$; $\bar{x}_0 = 1$, $\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = 1$; $\bar{x}_0 = 1$, $\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$ and $\bar{x}_0 = -1$, $\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}\right)_0 = -1$. Figure 6 shows comparison of results obtained by MPES and those obtained by IHBM in the case when supposed approximate solution includes only a constant term and the first harmonic (single harmonic balance method). As one can see, a good agreement of the results obtained by these two methods is achieved, but only when the assumption of dominance of primary resonance in the response is satisfied.

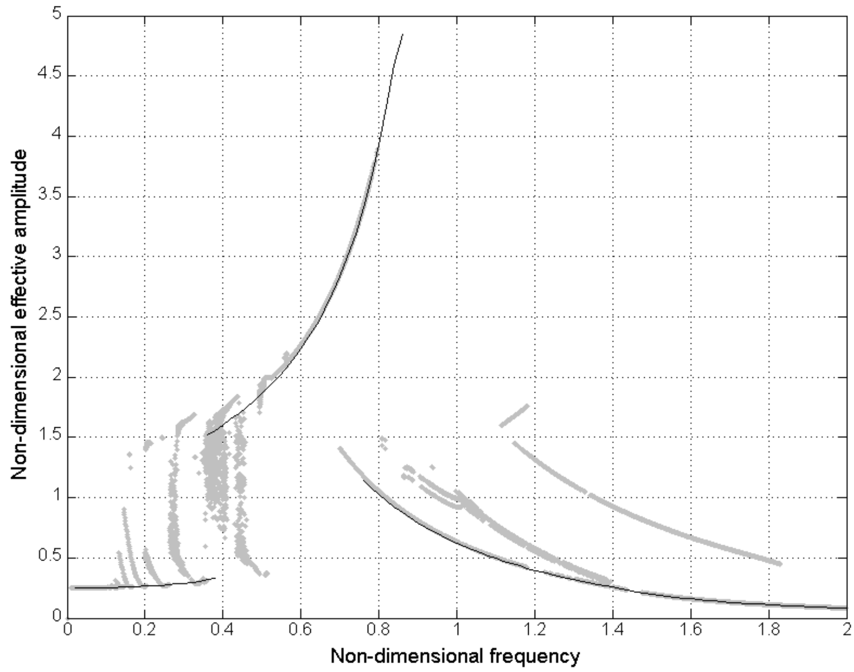


Figure 6. Comparison of the results obtained by MPES (dots) and by single harmonic balance method (line).

In Figure 7 the numerical results obtained by IHBM are compared with those obtained by MPES. Figure 7 shows excellent agreement between the results obtained by these methods. Non-periodic responses obtained by MPES are not found by the incremental harmonic balance method, because this method is limited only to consideration of periodic vibrations. Also, frequency response characteristics obtained by MPES are incomplete, because the results of MPES depend on given initial conditions, making it difficult to find all possible solutions.

As it is shown in [21], the accuracy of determining the eigenvalues of the monodromy matrix depend significantly on the number of harmonics included in the supposed approximate solution. Consequently, neglecting of very small harmonic terms of the actual time domain response can cause a very large error in evaluation of the eigenvalues of the monodromy matrix and can lead to incorrect prediction of the dynamical stability of the solution (Fig. 8). Figure 8(a) shows the relative differences of the effective amplitudes $\bar{x}_p$ (the a branch of the amplitude-frequency plot from Figure 7) obtained with N = 7, 12, 15, 18 and 30 harmonics with respect to the effective amplitudes $\bar{x}_p$ obtained with N = 100 harmonics

$$\left(\overline{x}_p\right)_{\mathrm{N\,diff}} = \frac{\left(\overline{x}_p\right)_{\mathrm{N}} - \left(\overline{x}_p\right)_{100}}{\left(\overline{x}_p\right)_{100}}, \qquad \mathrm{N} = 7,\ 12,\ 15,\ 18,\ 30\,. \tag{20}$$

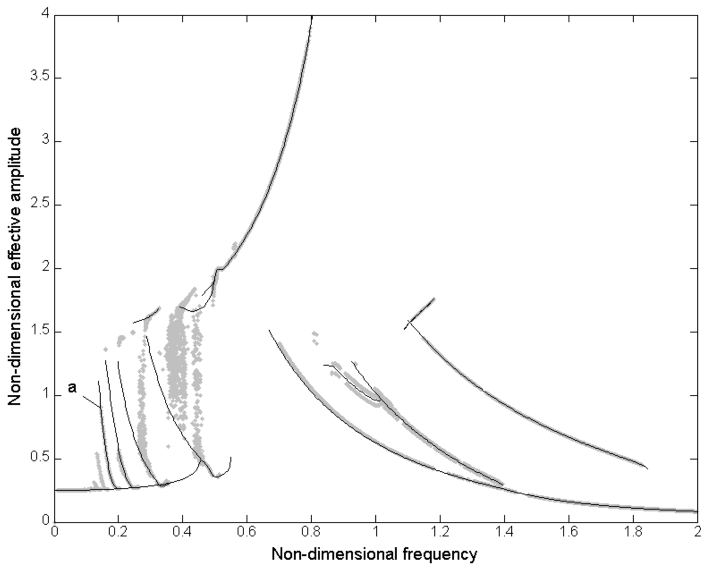
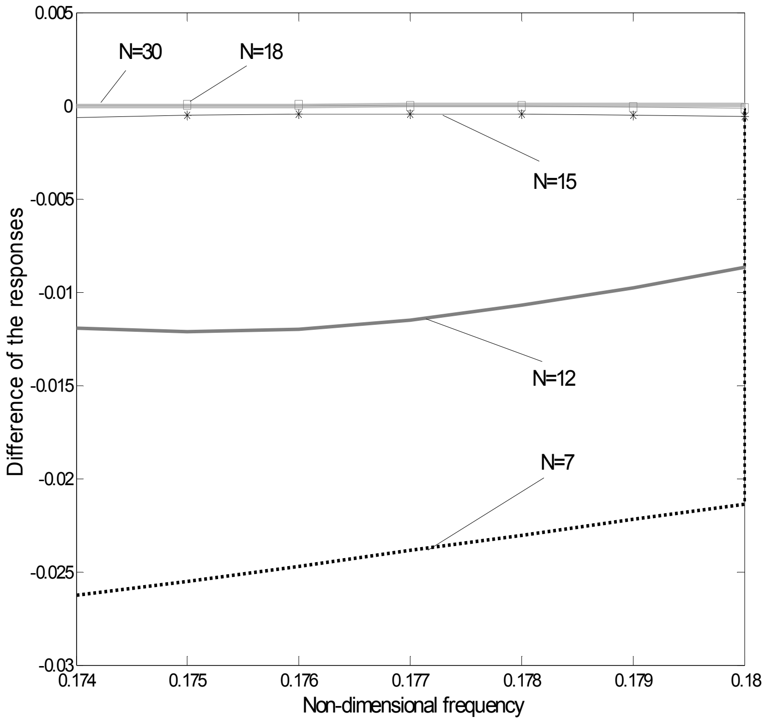


Figure 7. Comparison of the results obtained by MPES (dots) and by IHBM (line).

Figure 8(b) shows a corresponding plot of maximum modulus of the eigenvalues of the monodromy matrix $|\lambda_{\max}| = |\lambda_{\max}|(\eta, \mathrm{N})$. A very specific situation occurs at η = 0,176. In this case the value of $|\lambda_{\max}|$ is more precisely determined for N = 7 than for N = 12, N = 15 and N = 18, what can lead to incorrect estimation of dynamical stability of the response and can make bifurcation analysis difficult. The spectrum of the corresponding time domain response is shown in Fig. 9. In this example amplitudes of the harmonics for $\mathrm{N} > 8$ are exceptionally small in comparison to other terms of the spectrum and in-significantly influence effective amplitude $\overline{x}_p$ (amplitudes of the higher harmonics (N > 8) are less than 0.7 % of the amplitude of the dominant harmonic). Absence of convergence can be explained by essential difference between IHBM (the method used for obtaining approximate steady state solutions) and the procedure of evaluating the monodromy matrix (used for estimation of dynamical stability of the steady state solution by Floquet-Lyapunov theorem). Since the IHBM is based on the Galerkin's procedure, Fourier's coefficients of the supposed approximate solution ($a_0, a_\mathrm{i}, b_\mathrm{i},\ \mathrm{i} = 1,\dots,\mathrm{N}$) are determined in the way that differential equation (5) is satisfied in average, but not in every point of the response $\overline{x} = \overline{x}(\theta)$. On the other hand, stability estimation based on evaluation of the eigenvalues of the monodromy matrix depends only on the position of points $\theta_1, \theta_2, \dots, \theta_\mathrm{L}$ in which the system undergoes a stiffness change (Fig. 3) and estimation of stability is influenced only by the differences between the approximate and exact positions of the points $\theta_1, \theta_2, \dots, \theta_\mathrm{L}$. Increasing of number of harmonics N decreases average difference between the approximate and the exact solution and in this way increases the probability of more accurate determination of points $\theta_1, \theta_2, \dots, \theta_L$.

a)



b)

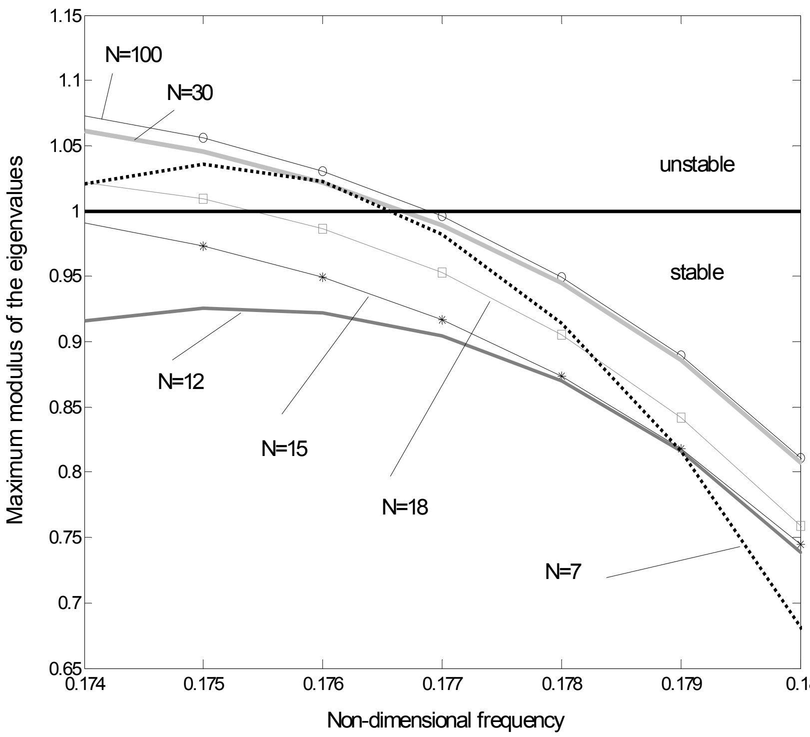


Figure 8. a) Differences of effective amplitudes $\left(\bar{x}_p\right)_{\mathrm{N\,diff}}$ (the a branch of the amplitude-frequency plot from Figure 7) obtained with N = 7, 12, 15, 18, and 30, and b) corresponding plot of maximum modulus of the eigenvalues of the monodromy matrix $|\lambda_{\max}| = |\lambda_{\max}|(\eta, \mathrm{N})$.

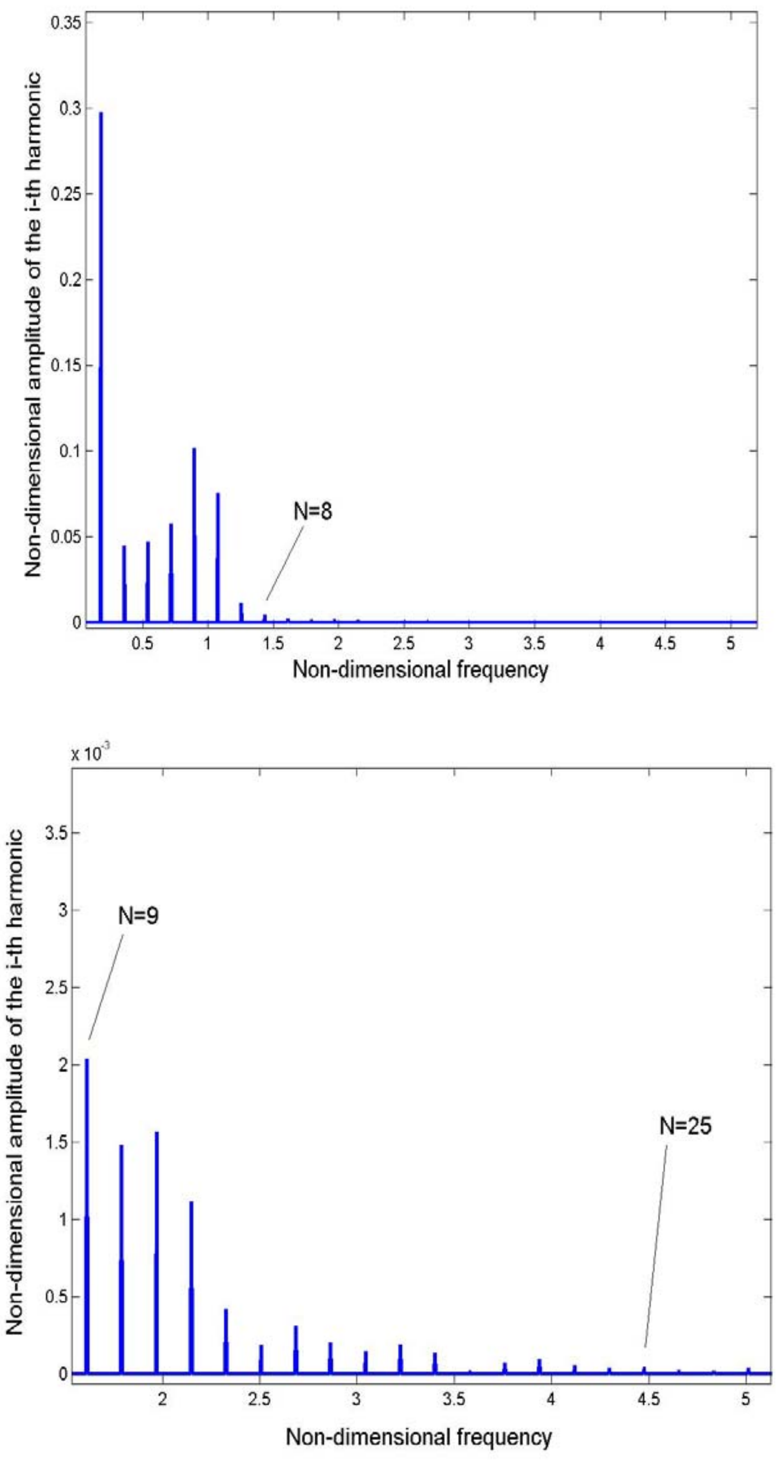


Figure 9. The spectrum of the time domain response for $\eta = 0{,}179$ (branch a in Fig 7).

CONCLUSIONS

Response and dynamical stability of oscillators with discontinuous or steep first derivative of restoring characteristic is considered in this paper. For that purpose, a simple single-degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic force excitation is analysed by the method of piecing the exact solutions (MPES) in the time domain and by the incremental harmonic balance method (IHBM) in the frequency domain. The stability of the periodic solutions obtained in the frequency domain by IHBM is estimated by the Floquet-Lyapunov theorem.

The considerable advantage of using this piecewise-linear model is in the possibility of expressing monodromy matrix exactly as a product of matrix exponentials, what is not possible for a general non-linear function. In this way, the inaccuracy of evaluating monodromy matrix can be caused only by insufficient precision of numerical determination of the times in which the system changes stage stiffness region, and by numerical procedures of evaluation matrix exponential and product of matrix exponentials. On the other hand, local solutions of differential equation (1) are known explicitly inside each of the stage stiffness, and can be repeatedly matched at $x = b$ and $x = -b$, to obtain a global solution of (1) in the time domain. Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in a closed form. But, the matching of local solutions can be numerically done very easily. Only approximation done by applying this procedure is in the precision of numerical determination of times in which the system changes stage stiffness region ($x = b$, $x = -b$).

Obtained frequency response characteristic is very complex and includes multi-frequency response for a single frequency excitation, jump phenomenon, multi-valued and non-periodic solutions. Determining of frequency response characteristic in the time domain by MPES is exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions, where many combinations of initial conditions have to be examined for obtaining complete frequency response characteristic. In the frequency domain, IHBM is very efficient and very well suited for obtaining wide range frequency response characteristics, parametric studies and bifurcation analysis. On the other hand, neglecting of very small harmonic terms (which in-significantly influence the r.m.s. values of the response and are very small in comparison to other terms of the spectrum) can cause very large error in evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution. Moreover, frequency ranges inside which the procedure of evaluation of eigenvalues of the monodromy matrix does not converge with increasing the number of harmonics included in the supposed approximate solution are detected.

ACKNOWLEDGMENTS

The data shown are result of a scientific project (Numerical and experimental research of non-linear mechanical systems) that was conducted with the support of the Ministry of science, education and sports, Republic of Croatia.

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ODZIV I DINAMIČKA STABILNOST VIBRACIJSKIH SUSTAVA S PREKINUTOM ILI STRMOM DERIVACIJOM POVRATNE KARAKTERISTIKE

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SAŽETAK

U radu su razmatrani odziv i dinamička stabilnost vibracijskih sustava koji imaju prekinutu ili strmu prvu derivaciju povratne karakteristike. U tu svrhu je analiziran jednostavni vibracijski sustav s jednim stupnjem slobode gibanja i karakteristikom krutosti koja se sastoji od linearnih segmenata koji je uzbuđen s harmonijskom silom. Odziv sustava je u vremenskoj domeni dobiven s metodom povezivanja egzaktnih rješenja po segmentima (MPES), a u frekvencijskoj domeni s inkrementalnom metodom harmonijske ravnoteže (IHBM). Procjena stabilnosti periodičnih rješenja dobivenih u frekvencijskoj domeni korištenjem IHBM izvršena je primjenom Floquet-Lyapunovog teorema. Dobiveni graf funkcije povećanja je vrlo složen i sadrži višefrekvencijske odzive uzrokovane jednofrekvencijskom uzbudom, tzv. skokove amplitude, te višestruka i neperiodična rješenja. Određivanje grafa funkcije povećanja u vremenskoj domeni s MPES izuzetno je dugotrajno a to je najizraženije u područjima frekvencija u kojima postoje višestruka stabilna rješenja. IHBM, s kojom se odziv sustava određuje u frekvencijskoj domeni vrlo je efikasna i dobro prilagođena metoda za određivanje cjelovitog grafa funkcije povećanja, kao i za parametarsku i bifurkacijsku analizu. S druge strane, zanemarivanje vrlo malih harmonika (koji neznatno utječu na srednju vrijednost i efektivnu amplitudu odziva i koji su vrlo mali u odnosu na ostale harmonike u spektru) može uzrokovati vrlo velike pogreške u određivanju vlastitih vrijednosti prijenosne matrice i tako dovesti do pogrešne procjene dinamičke stabilnosti rješenja. Štoviše, uočena su frekvencijska područja unutar kojih postupak određivanja vlastitih vrijednosti prijenosne matrice ne konvergira s povećanjem broja harmonika uključenih u pretpostavljeno približno rješenje.

KLJUČNE RIJEČI

dinamička stabilnost, graf funkcije povećanja, nelinearne vibracije, sustav linearan po segmentima

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