Interdisciplinary Description of Complex Systems 22(3), 276-295, 2024

GRADED LOGICS

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DOI: 10.7906/indecs.22.3.4
Regular article

Received: 21 May 2024.
Accepted: 15 June 2024.

ABSTRACT

This article reviews graded logics. A brief overview of various kinds of logics, classical, fuzzy and graded, is given. Their role in modelling reasoning and differences between logics are discussed. Graded logic specifically is discussed. Possible and some realised applications of graded logic are pointed out. Conclusions about relations between considered logics are given.

KEY WORDS

artificial intelligence, logics, graded logics, fuzzy logic, aggregation

CLASSIFICATION

JEL: O30

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INTRODUCTION

The intelligence of humans is achieved by reflex mechanisms and by processes of reasoning that operate on internal representations (models) of knowledge [1, 2].

Logic is one of the tools for modelling the observable properties of human reasoning. It is the study of correct reasoning. It includes both formal and informal logic. Formal logic is the study of deductively valid inferences or logical truths. Human communication is based on natural languages and consists of linguistic sentences. Some sentences are truth-bearers, they can be true or false (or partially true and partially false). In logic, such sentences are called propositions or statements. Formal logic deals with premises and inference rules. A premise is a true or false declarative statement (a proposition) used in an inference rule to prove the truth of another proposition called a conclusion. Inference rules consist of a set of premises and a conclusion. An inference rule examines how the conclusion follows from the premises due to the structure of the inference rule alone, independent of the topic and content of the premises, and the rule uses formal language So, the term 'logic' in this paper refers to a logical formal system that articulates a reasoning system.

In the paper, a brief overview of various kinds of logic, classical (Boolean), fuzzy (Zadeh's) and graded (Dujmović's), is given, and relationships between them are discussed. Relationships between the classical logic, the fuzzy logic and the graded logic have been discussed in [4], as well as in [9]. Dujmović's results (beginning from [6], and later) are a strong contribution to the development and generalization of aggregation not only as part of fuzzy logic and soft computing. Those results, [6] to its current status [7, 8], improve Zadeh's approach in dealing with uncertain and vague information common in human reasoning. Relationships between the classical Boolean logic (BL), the graded logic (GL) and the fuzzy logic (FL) have been broadly discussed in [7] and it has been shown that those logics are subset-structured as follows: BL $\subset$ GL $\subset$ FL. BL is primarily a crisp bivalent propositional calculus. GL includes BL plus graded truth, graded idempotent conjunction/disjunction, weight-based semantics and (less frequently) nonidempotent hyperconjunction/hyperdisjunction, among other notions. FL includes GL, various forms of nonidempotent conjunction/disjunction, and other generalisations of multivalued logic.

The paper highlights the role of considered logics in modelling reasoning and the differences between them. In Section 2 classical logic is considered. Section 3 deals with fuzzy logic. In Section 4 graded logic is described. Possible and some realised applications of graded logic, described in the literature, are mentioned. Conclusions about relationships between considered logics are given in Section 5. A list of references is provided.

CLASSICAL LOGIC

Logic [1, 2, 7], has its origins in ancient Greek philosophy and mathematics. The first known systematic study of logic was carried out by Aristotle (Organon). Stoics took five basic inference rules (chains of conclusions that lead to the desired goal) as valid without proof, including the rule we now call Modus Ponens (Latin for: a mode that affirms):

$$\frac{\alpha \Rightarrow \beta,\ \alpha}{\beta}$$

whenever sentences of the form '$\alpha \Rightarrow \beta$' (implication, if – then statement: α implies β), and 'α' are given, then the sentence 'β' can be inferred.

George Boole introduced the first comprehensive and workable system of formal logic (Boole, G. *The mathematical analysis of logic*, 1847). Boole's logic was closely modelled on the ordinary algebra of real numbers and used substitution of logically equivalent expressions as its primary inference method.

That logic is nowadays known as classical bivalent propositional calculus with crisp truth values (from the set {0, 1}) formalised as Boolean algebra. In propositional logic (calculus) methods for correct use of propositions are studied. In general, propositions can be crisp or graded, depending on the type of truth value they bear. Declarative sentences expressing assertions that are either completely true (coded as 1) or completely false (coded as 0) are called crisp propositions or statements. Classical logic (a propositional calculus from Aristotle to G. Boole) deals with crisp propositions only: the only logic values are true (numerically coded as 1) and false (numerically coded as 0), and all logic variables belong to the set {0, 1}.

The basic logic functions are the pure conjunction (and) $z = x \wedge y = \min(x, y)$, the pure disjunction (or) $z = x \vee y = \max(x, y)$ and negation $\bar{x} = 1 - x$. Obviously, $\bar{\bar{x}} = x$ (involution), $x \vee \bar{x} = 1$, $x \wedge \bar{x} = 0$, $x \wedge x = x \vee x = x$. Under these assumptions a Boolean function of n variables $z = f(x_1, x_2, \ldots, x_n)$ ($x_i \in \{0,1\}$ is defined using 2^n combinations (variations) of input values (from 00 … 0, to 11 …1): [Let A is a set of elements $A = \{a_1, a_2, \ldots, a_k\}$. The variations with repetition of k elements taken n by n are the arranged groups formed by n elements from A (which may be repeated). This is represented as $\bar{V}_k^n = k \cdot k \cdot \ldots \cdot k = k^n$. For $k = 2$, it is 2^n.] Consequently, there are 2^{2^n} different Boolean functions of n variables.

If $n = 1$, there are four different Boolean functions of one element, as shown in Table 1.

Table 1. Boolean functions $y = f(x)$.

x	0	1	Function	Name
y_0	0	0	$y_0 = 0$	Constant 0
y_1	0	1	$y_1 = x$	Variable x
y_2	1	0	$y_2 = \bar{x}$	Negation
y_3	1	1	$y_3 = 1$	Constant 1

If $n = 2$, there are 16 different Boolean functions of n elements, as shown in Table 2.

Pure conjunction and disjunction are idempotent (2.1), commutative (2.2), associative (2.3) and distributive (2.4), have neutral elements (2.5), annihilators (2.6), and inverse elements (2.7):

$$x \wedge x = x, \qquad x \vee x = x, \tag{2.1}$$

$$x \wedge y = y \wedge x, \qquad x \vee y = y \vee x, \tag{2.2}$$

$$x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z), \qquad x \vee y \vee z = (x \vee y) \vee z = x \vee (y \vee z), \tag{2.3}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \tag{2.4}$$

$$x \wedge 1 = x, \qquad x \vee 0 = x, \tag{2.5}$$

$$x \wedge 0 = 0, \qquad x \vee 1 = 1, \tag{2.6}$$

$$x \wedge \bar{x} = 0, \qquad x \vee \bar{x} = 1. \tag{2.7}$$

The negation of Boolean functions is based on De Morgan's laws: $\overline{x \wedge y} = \bar{x} \vee \bar{y}$, $\overline{x \vee y} = \bar{x} \wedge \bar{y}$. De Morgan's laws (De Morgan, A. *Formal logic or the calculus of inference necessary and probable*, London: Taylor and Walton, 1847) show the duality of conjunction and disjunction: if we have one of these operations, the other one can be obtained as a mirrored dual operation.

To make a conjunction from a disjunction we use $x \wedge y = \overline{\bar{x} \vee \bar{y}}$, and to make a disjunction from a conjunction we use $x \vee y = \overline{\bar{x} \wedge \bar{y}}$. De Morgan's law can be written for general Boolean function f: $\{0, 1\}^n \rightarrow \{0, 1\}$ as follows:

$$\overline{f(x_1, \ldots, x_n, \wedge, \vee, 0, 1)} = f(\overline{x_1}, \ldots, \overline{x_n}, \vee, \wedge, 1, 0).$$

Table 2. Boolean functions $z = f(x, y)$.

x	0	1	0	1	Function	Name
y	0	0	1	1		
z_0	0	0	0	0	$z_0 = 0$	Constant 0
z_1	1	0	0	0	$z_1 = x \downarrow y = \bar{x} \wedge \bar{y} = \overline{x \vee y}$	Nor
z_2	0	1	0	0	$z_2 = x \wedge \bar{y} = \overline{x \longrightarrow y}$	Abjunction
z_3	1	1	0	0	$z_3 = \bar{y}$	Negation of y
z_4	0	0	1	0	$z_4 = \bar{x} \wedge y = y \to x$	Abjunction
z_5	1	0	1	0	$z_5 = \bar{x}$	Negation of x
z_6	0	1	1	0	$z_6 = x \oplus y = (x \wedge \bar{y}) \vee (\bar{x} \wedge y) = \overline{x \sim y}$	Exclusive or
z_7	1	1	1	0	$z_7 = x \mid y = \bar{x} \vee \bar{y} = \overline{x \wedge y}$	Nand
z_8	0	0	0	1	$z_8 = x \wedge y$	Conjunction (and)
z_9	1	0	0	1	$z_9 = x \sim y = (\bar{x} \wedge \bar{y}) \vee (x \wedge y) = \overline{x \oplus y}$	Equivalence
z_{10}	0	1	0	1	$z_{10} = (x \wedge \bar{y}) \vee (x \wedge y) = x \wedge (\bar{y} \vee y) = x$	Absorption of y
z_{11}	1	1	0	1	$z_{11} = y \to x = \overline{y \wedge \bar{x}} = \bar{y} \vee x$	Implication
z_{12}	0	0	1	1	$z_{12} = (\bar{x} \wedge y) \vee (x \wedge y) = y \wedge (\bar{x} \vee x) = y$	Absorption of x
z_{13}	1	0	1	1	$z_{13} = x \to y = \overline{x \wedge \bar{y}} = \bar{x} \wedge y$	Implication
z_{14}	0	1	1	1	$z_{14} = x \vee y$	Disjunction (or)
z_{15}	1	1	1	1	$z_{15} = 1$	Constant 1

Classical Boolean logic can be derived in a deductive axiomatic way as Boolean algebra using a set with two elements $B = \{0, 1\}$ and binary internal operations $\wedge$ and $\vee$:

$$\forall x,y \in B \Rightarrow x \wedge y \in B,\ x \vee y \in B,$$

using the following three axioms:

A1. Binary operations $\wedge$ and $\vee$ are commutative and distributive:

$\forall x,y \in B$:

$x \wedge y = y \wedge x,$ $\qquad$ $x \vee y = y \vee x,$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$ $\qquad$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$

A2. On set B binary internal operations have two different neutral elements:

$(\forall x \in B)\ (\exists 0 \in B) \Rightarrow x \vee 0 = x,$

$(\forall x \in B)\ (\exists 1 \in B) \Rightarrow x \wedge 1 = x.$

A3. On set B each element x has a unique inverse element:

$(\forall x \in B)\ (\exists \bar{x} \in B) \Rightarrow x \vee \bar{x} = 1,$

$(\forall x \in B)\ (\exists \bar{x} \in B) \Rightarrow x \wedge \bar{x} = 0.$

Using these axioms, it is possible to prove various properties (idempotency, De Morgan's laws, etc.).

In classical Boolean logic, simultaneity and substitutability are not graded. Conjunction is the only model of simultaneity and disjunction is the only model of substitutability. Conjunction and disjunction are dual based on the De Morgan's laws:

$x \wedge y = \overline{\overline{(x \wedge y)}} = \overline{\bar{x} \vee \bar{y}} = 1 - (1 - x) \vee (1 - (1 - y)),$

$x \vee y = \overline{\overline{(x \vee y)}} = \overline{\bar{x} \wedge \bar{y}} = 1 - (1 - x) \wedge (1 - (1 - y)).$

Tautologies are defined as formulas that are always true. Following are some tautologies that are frequently used in bivalent logic reasoning.

The law of the excluded middle: $x \vee \bar{x} = 1$.

Modus Ponens (if x is satisfied and x implies y, then that implies that y is also satisfied):

$$(x \wedge (x \rightarrow y)) \rightarrow y = (x \wedge (\bar{x} \vee y)) \longrightarrow y = (x \wedge y) \longrightarrow y = \bar{x} \vee \bar{y} \vee y) = \bar{x} \vee 1 = 1.$$

It was said that classical Boolean functions of n variables are defined only in 2^n isolated vertices $\{0, 1\}^n$ of the unit hypercube $[0, 1]^n$. For $n = 3$, there are 2^3 vertices, Figure 1.

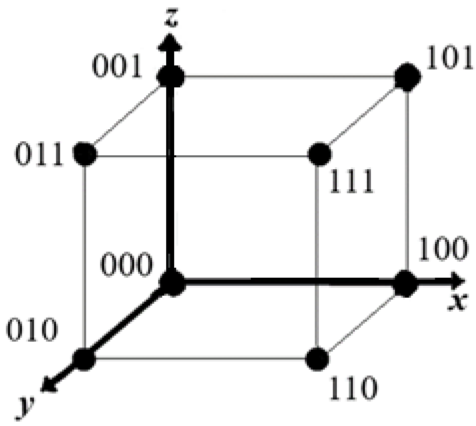

Figure 1. Boolean function f domain for $n = 3$.

Classical logic was the basis for the realization of many applications in the field of artificial intelligence [1].

FUZZY LOGIC

Jan Lukasiewicz and his school developed logics with intermediate truth values and stimulated considerable interest in multivalued logics during the 1930s (Łukasievicz J.: *O logice trójwartościowej, Ruch Filozoficzny* 5, 169-171, 1920 – in English: *On Three-Valued Logic*, McCall, S., ed., Polish Logic 1920-1939, Claredon Press, Oxford, 1967).

Most concepts in human reasoning (such as truth, importance, suitability, simultaneity, etc.) are a matter of degree. Consequently, some areas of human logic reasoning are not reducible to zeroes and ones, and it cannot be modelled only in the vertices of the hypercube $\{0, 1\}^n$ as in the case of classical bivalent Boolean logic. Since the truth is a matter of degree, it belongs to interval $I = [0, 1]$, and all humanized models of logic reasoning must be applicable everywhere inside the hypercube $[0, 1]^n$, Figure 2.

In 1965 Lotfi Zadeh introduced fuzzy sets [3], the first successful step toward the wide use of graded concepts in science, computing, and engineering. Important concepts of fuzzy logic (linguistic variables, the calculus of fuzzy if-then rules), were also introduced by Zadeh. For more information about that and about references, [4] can be consulted, here is a short overview.

FUZZY SETS

Fuzzy logic is an extension of Boolean logic based on the mathematical theory of fuzzy sets, which is a generalization of the classical set theory. Just as there is a connection between the classical (Boolean) logic and the classical (Cantorian) notion of a set, there is a connection between fuzzy logic and fuzzy set theory. By introducing the notion of degree in the verification of a condition, thus enabling a condition to be in a state other than true or false, fuzzy logic provides very valuable flexibility for reasoning, which makes it possible to take into account inaccuracies and uncertainties.

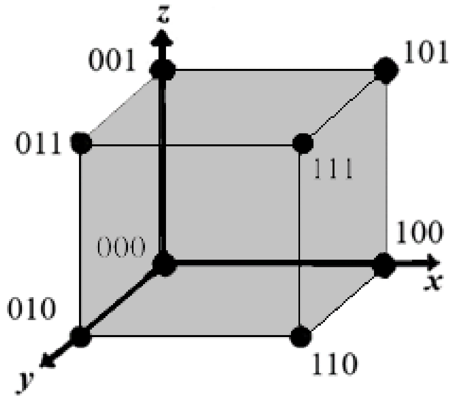

Figure 2. Function *f* domain for $n = 3$, $x,y,z \in [0, 1]$, for graded logics, fuzzy and graded (in the narrow sense).

One advantage of fuzzy logic to formalize human reasoning is that the inference rules are set in natural language [4, 5]. For example, here are some rules of conduct that a guest of a restaurant follows with the specific objective of deciding the amount of a tip at the end of a meal, depending on the quality of the food and the quality of service:

If the quality of the food is awful … and if the quality of service is poor … then the tip is low.
If the quality of the food is awful … and if the quality of service is good ... then the tip is low.
If the quality of the food is awful… and if the quality of service is excellent ... then the tip is medium.
If the quality of the food is delicious… and if the quality of service is good ... then the tip is high.
If the quality of the food is delicious… and if the quality of service is excellent ... then the tip is high.

Intuitively, it thus seems that the input variables like in this example are approximately appreciated by the brain, such as the degree of verification of a condition in fuzzy logic.

To exemplify each definition of fuzzy logic, we will consider some elements of a fuzzy inference system whose specific objective is to apply some rules of conduct that a driver follows in front of a traffic light, assuming that he does not want to lose his driver's licence.

Saying that the theory of fuzzy sets is a generalization of the classical set theory means that the latter is a special case of fuzzy sets theory, the classical set theory is a subset of the theory of fuzzy sets, as Figure 3 illustrates.

Fuzzy logic is based on fuzzy set theory, which is a generalization of the classical set theory. Following the habits of the literature, we will use the terms fuzzy sets instead of fuzzy subsets. The classical sets are also called clear sets, as opposed to vague, and by the same token classical (Boolean) logic is also known as binary (bivalent).

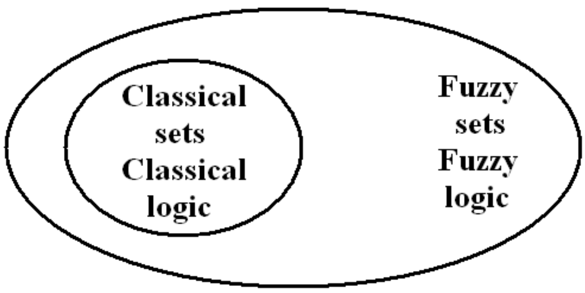

Figure 3. The classical set theory is a subset of the theory of fuzzy sets.

Figure 4 shows the membership function chosen to characterize the subset of “average” speeds.

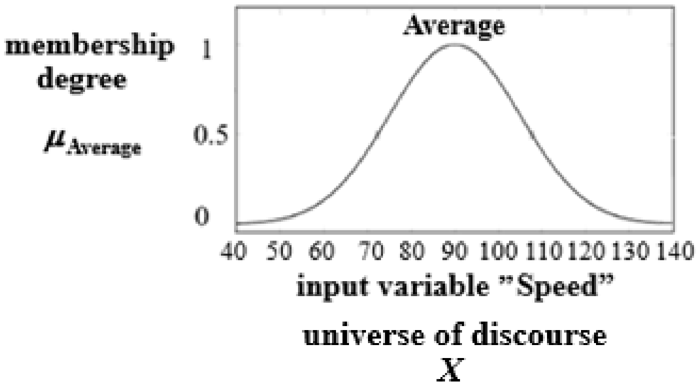

Figure 4. Membership function characterizing the subset of "average" speed.

Definition 1. Let X be a set. A fuzzy subset A of X is characterized by a *membership function f*: $X \to [0; 1]$. Note: This membership function is equivalent to the identity function of a classical set.

In our traffic light example, we will redefine membership functions for each fuzzy set of each of the following three variables.

- Input 1: speed. Subsets: low, medium and high;
- Input 2: distance (from the traffic light). Subsets: close and far;
- Output: braking (pressure). Subsets: mild, average and hard.

The shape of the membership function is chosen arbitrarily by following the advice of the expert or by statistical studies: sigmoid, hyperbolic, tangent, exponential, Gaussian or any other form can be used.

Figure 5 shows the difference between a conventional set and a fuzzy set.

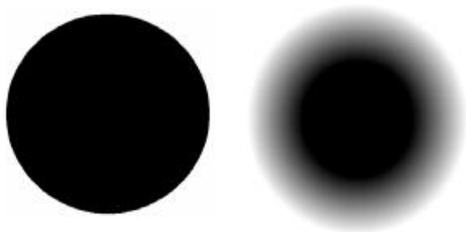

Figure 5. Graphical representation of a conventional set and a fuzzy set.

To define the characteristics of fuzzy sets, we are redefining and expanding the usual characteristics of classical sets.

Fuzzy sets have some properties. Here are definitions of the most important properties.

Let X be a set (also known as *universe of discourse*), A fuzzy subset of X and μ_A the membership function characterizing fuzzy set A. Function $\mu_A(x)$ is called the *membership degree* of x, $x \in X$, in A.

The set X can be, for example, the set of positive real numbers, the quality of the food at a restaurant on the subjective scale from 0 to 10, speeds (in km/h), set of braking pressures (here in unnamed subjective numbers on the scale from 0 to 30, etc.).

In Figure 6 comparison of the two membership (the indicator and the membership) functions corresponding to the previous sets is given.

Definition 2. The *height* of A, denoted $h(A)$, corresponds to the upper bound of the codomain of its membership function: $h(A) = \sup \{\mu_A(x) \mid x \in X\}$.

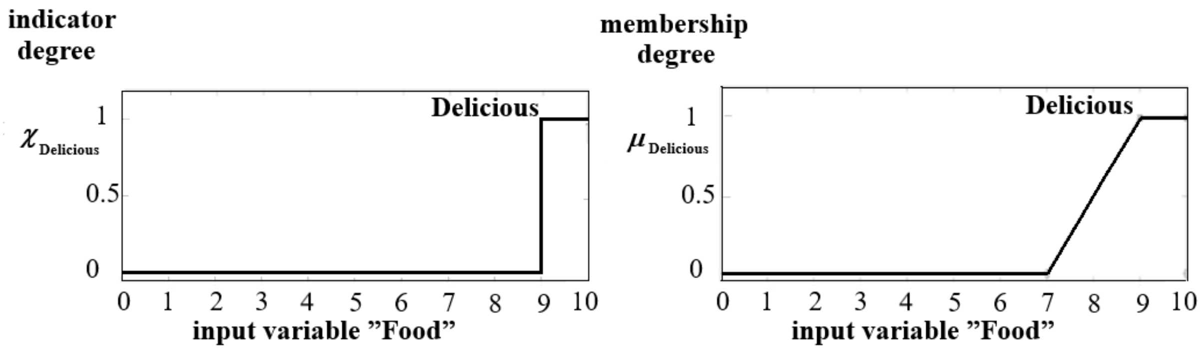

Figure 6. Comparison between an indicator (characteristic) function of a conventional set (left) and a membership function of fuzzy set (right).

Definition 3. Fuzzy set A is said to be *normalised* if and only if $h(A) = 1$. In practice, it is extremely rare to work on non-normalised fuzzy sets.

Definition 4. The *support* of fuzzy set A is the set of elements of X belonging to at least some A (i.e. the membership degree of x is strictly positive). In other words, the support is the set

$$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 5. The *kernel* of fuzzy set A is the set of elements of X belonging entirely to A. In other words,

$$\text{kernel}(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

By construction, $\text{kernel}(A) \subseteq \text{supp}(A)$.

Definition 6. An *α-cut* of a fuzzy set A is the classical subset of elements with a membership degree greater than or equal to α

$$\alpha\text{-cut}(A) = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Another membership function for a variable ‘braking’ (X) (in unnamed (subjective) units on the scale from 0 to 30) through which we have included the above properties is presented in Figure 7.

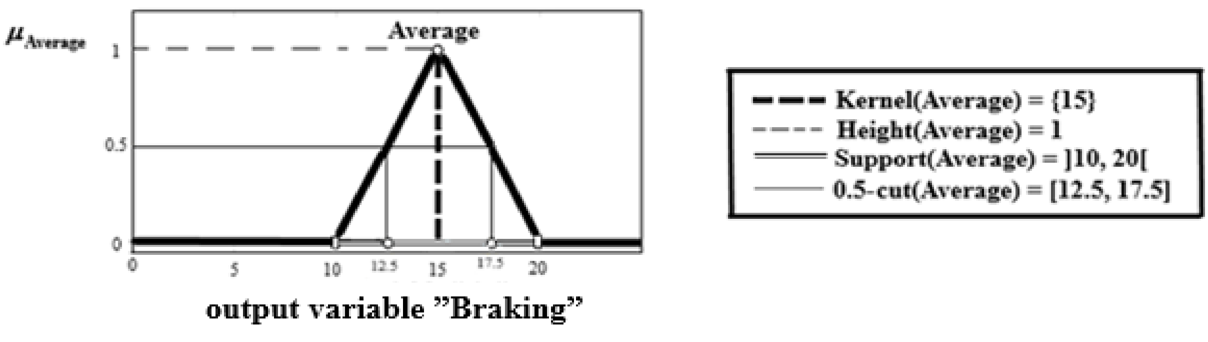

Figure 7. A membership function μ_{Average} with properties displayed.

We can see that if A was a conventional set, we would simply have

$$\text{supp}(A) = \text{kernel}(A) \text{ and } h(A) = 1, (h(A) = 0, A = \emptyset).$$

Our definitions can therefore recover the usual properties of classical sets.

THE LINGUISTIC VARIABLES

The concept of the membership function discussed above allows us to define fuzzy systems in natural language, as the membership function couples fuzzy logic with linguistic variables that we will define in the sequel.

Definition 7. Let *V* be a variable (speed, braking, etc.), *X* the range of values of the variable (universe of discourse), and T_V a finite or infinite set of fuzzy sets. A *linguistic variable* corresponds to the triplet (V, X; T_V).

Here are some examples of linguistic variables, in Figures 8-10.

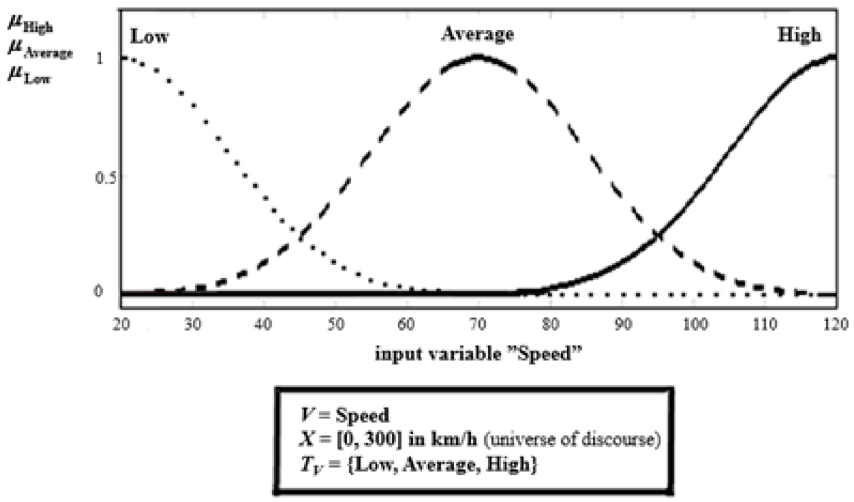

Figure 8. Linguistic variable "Speed".

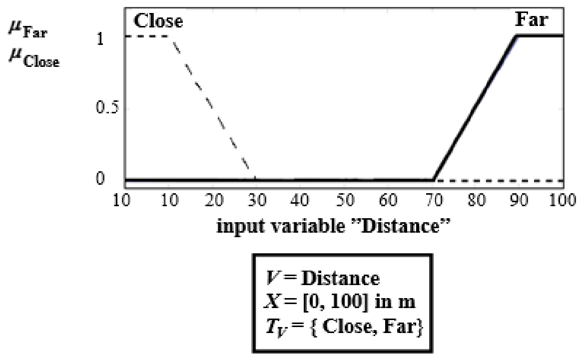

Figure 9. Linguistic variable "Distance" (from the traffic light).

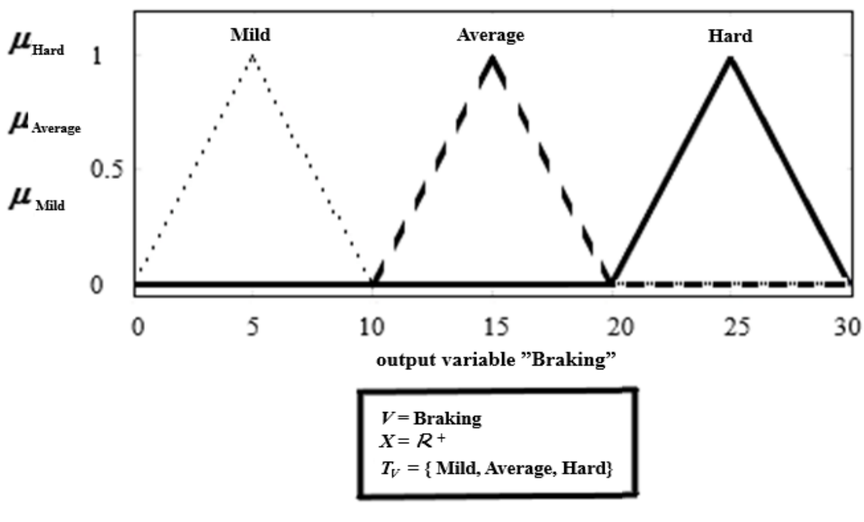

Figure 10. Linguistic variable "Braking" (in unnamed subjective numbers on the scale from 0 to 30, i.e. in positive real numbers).

When we define the fuzzy sets of linguistic variables, the goal is not to exhaustively define the linguistic variables. Instead, we only define a few fuzzy (sub)sets that will be useful later in the definition of the rules that we apply. This is for example the reason why we have not defined the subset "middle" for the distance from the traffic lights. Indeed, this subset could not be useful in our rules. Similarly, it is also the reason why (for example) 30 represents a harder braking pressure than 25, while 25 however belongs more to the fuzzy set "hard" than 30: this is because 30 is seen not as hard but very hard (or exorbitant if you want to change adjective). However, we have not created the fuzzy set "very hard" because we do not need it in our rules.

THE FUZZY OPERATORS

To easily manipulate fuzzy sets, we are redefining the operators of the classical set theory to fit the specific membership functions of fuzzy logic for values strictly between 0 and 1. Unlike the definitions of the properties of fuzzy sets that are always the same, the definition of operators on fuzzy sets is chosen, like membership functions. Here are the two sets of operators for the complement (NOT), the intersection (AND) and the union (OR) most commonly used, Table 3.

Table 3. Most commonly used fuzzy operators.

Name	Intersection AND $\mu_{A\cap B}(x)$	Union OR $\mu_{A\cup B}(x)$	Complement NOT $\mu_{\bar{A}}(x)$
Zadeh operators min/max	$\min(\mu_A(x), \mu_B(x))$	$\max(\mu_A(x), \mu_B(x))$	$1 - \mu_A(x)$
Probabilistic prod/probor	$\mu_A(x)\times\mu_B(x)$	$\mu_A(x) + \mu_B(x) - \mu_A(x)\times\mu_B(x)$	$1 - \mu_A(x)$

With the usual definitions of fuzzy operators, we always find the properties of commutativity, distributivity, and associativity classics. However, there are two notable exceptions:

- in fuzzy logic, the law of excluded middle is contradicted: $A\cup\bar{A} \neq X$, i.e. $\mu_{A\cup\bar{A}}(x) \neq 1$;
- in fuzzy logic, an element can belong to A and not A at the same time:

$$A \cap \bar{A} \neq \emptyset, \text{ i.e. } \mu_{A\cap\bar{A}}(x) \neq 0.$$

Note that these elements correspond to the set supp(A) – kernel(A).

REASONING IN FUZZY LOGIC

As we know, in classical logic, the inference rules may be of the form (already mentioned here):

$$\left\{\begin{array}{c} If\ p\ then\ q, p\ true \\ then\ q\ true \end{array}\right.$$

In fuzzy logic, fuzzy reasoning, also known as approximate reasoning, is based on fuzzy rules that are expressed in natural language using linguistic variables which we have given the definition previously. A fuzzy rule has the form:

$$If\ x \in A\ and\ y \in B\ then\ z \in C,$$

with A, B and C fuzzy sets. For example:

'*If* (*the distance is close*), *then* (*the braking is hard*)'.

The variable "braking" belongs to the fuzzy set "hard" to a degree that depends on the degree of validity of the premise, i.e. the membership degree of the variable "distance" to the fuzzy set "close". The underlying idea is that the more propositions in the premise are checked, the more the suggested output actions must be applied. To determine the degree of truth of the fuzzy proposition “braking will be hard”, we must define the fuzzy implication.

Like other fuzzy operators, there is no single definition of the fuzzy implication: the fuzzy system designer must choose among the wide choice of fuzzy implications already defined, or set it by hand. Table 4 contains four definitions of fuzzy implication most commonly used.

Table 4. Some fuzzy implications.

Name	Truth value
Mamdani	$min(f_a(x), f_b(x))$
Larsen	$f_a(x) \times f_b(x)$
Zadeh	$max(1 - f_a(x), min(f_a(x), f_b(x))$
Kleene-Dienes	$max(1 - f_a(x), f_b(x))$

Notably, not all of these four implications generalize the classical implication. There are other definitions of fuzzy implication generalizing the classical implication, but are less commonly used.

If we choose the Mamdani implication, here is what we get for the fuzzy rule

'*If* (*the distance is close*), *then* (*braking is hard*)',

where the distance is rated 18.7 (m) (if the rating is done by an autonomous car pilot in a self-driving car), a graphical interpretation of obtaining the conclusion is presented in Figure 11.

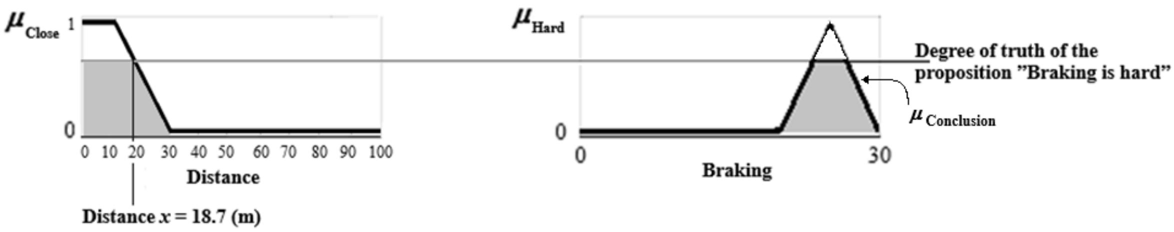

Figure 11. Example of fuzzy implication, fuzzy sets “Hard” and “Conclusion” are depicted by their membership degrees.

Similarly, [4], the conclusion would be obtained by rating ’approximately 20 m’, (if it is done by a driver trying not to lose his driver's license).

Thus, fuzzy logic allows building inference systems in which decisions are without discontinuities, flexible, and nonlinear, i.e. closer to human behaviour than classical logic is. The fuzzy logic theory offers a multitude of connectives that can be used as aggregators to aggregate membership values representing uncertain information. In the case of Zadeh's operators, *min* for conjunction and *max* for disjunction, used as aggregators, only inputs with extreme values affect the value of the output fuzzy set. However, both intuitive and formal criteria of human reasoning contain numerous requirements that are combined using models of simultaneity and substitutability (partial conjunction and partial disjunction), which set requirements for further development of graded aggregators.

GRADED LOGIC

Aristotle, Boole, De Morgan, and others developed bivalent classical logic, Łukasievicz and others dealt with trivalent and multivalent logics, Zadeh and others developed *n*-valent fuzzy logic, and Dujmović developed graded logic, as a continuous generalization of classical bivalent Boolean logic. All variables belong to the unit interval $I = [0, 1]$, all logical phenomena and their models appearing in graded logic also occur inside the unit hypercube I^n, where $n > 1$ (refer to Figure 2). Therefore, graded logic is continuous.

The statement "Vučević is a tall man" is not crisp because it is neither completely true nor completely false. Of course, those who know basketball will agree that it is truer than the statement "Ivanović is a tall man". These statements assert the value of the evaluated person and their degree of truth is located between true and false. Such statements (graded propositions) are called *value statements*. The degree of truth of a value statement is a human percept, interpreted as the degree of satisfaction of requirements. Truth comes in degrees. Graded propositions use a degree of truth that can be continuously adjustable from false to true in the interval [0, 1]. Such a degree of truth can also be interpreted as the degree of membership in a fuzzy set where the full membership corresponds to the degree of truth 1, and no membership corresponds to the degree of truth 0.

Graded logic (GL) uses graded truth and processes it using graded aggregators.

The main concepts in GL are means and aggregation, [7].

If we have n real numbers $x_1, \dots, x_{\mathrm{n}}$, $n > 1$, the mean value of these numbers is a value $M(x_1, \dots, x_{\mathrm{n}})$, located somewhere between the smallest and the largest of the numbers:

$$\min(x_1, \dots, x_{\mathrm{n}}) \leq M(x_1, \dots, x_{\mathrm{n}}) \leq \max(x_1, \dots, x_{\mathrm{n}}) \,. \qquad (4.1)$$

This property of function is called *internality*. In GL $x_1, \dots, x_{\mathrm{n}}$ are degrees of truth, and they belong to the unit interval $I = [0, 1]$: $x_i \in I$, $i = 1, \dots, n$, and $M: I^n \rightarrow I$.

Relation (4.1) can be rewritten as follows:

$$AND = x_1 \wedge \dots \wedge x_{\mathrm{n}} \leq M(x_1, \dots, x_{\mathrm{n}}) = x_1 \vee \dots \vee x_{\mathrm{n}} = OR. \qquad (4.1a)$$

In GL, means are logic functions, modelling observable properties of human reasoning.

Relation (4.1) indicates that the mean M (as a logic function) can be linearly interpolated between AND and OR as follows:

$$M(x_1, \dots, x_{\mathrm{n}}) = (1 - \omega)(x_1 \wedge \dots \wedge x_{\mathrm{n}}) + \omega(x_1 \vee \dots \vee x_{\mathrm{n}}),\ 0 \leq \omega \leq 1. \qquad (4.3)$$

Parameter $\omega \in I$ defines the location of M in the space between conjunction and disjunction, or proximity of M to disjunction, and it is called *orness* (disjunction degree). From (4.3) orness of M is:

$$\omega = \frac{M(x_1, \dots, x_{\mathrm{n}}) - (x_1 \wedge \dots \wedge x_{\mathrm{n}})}{(x_1 \vee \dots \vee x_{\mathrm{n}}) - (x_1 \wedge \dots \wedge x_{\mathrm{n}})} \,. \qquad (4.4)$$

Similarly, parameter $\alpha \in I$ which defines the location of M in the space between conjunction and disjunction, or proximity of M to conjunction, *andness* (conjunction degree), is:

$$\alpha = \frac{(x_1 \wedge \dots \wedge x_{\mathrm{n}}) - M(x_1, \dots, x_{\mathrm{n}})}{(x_1 \vee \dots \vee x_{\mathrm{n}}) - (x_1 \wedge \dots \wedge x_{\mathrm{n}})} \,. \qquad (4.4a)$$

Relations (4.3) and (4.4) indicate that each mean could be interpreted as a mix of disjunctive and conjunctive properties. In GL parameterized means are of particular interest. Such means have adjustable parameters $r(\omega)$ (or $r(\alpha)$), that can be used to adjust the logical properties of means and provide a continuous transition from *AND* to *OR*:

$$\begin{gathered} AND = x_1 \wedge \dots \wedge x_{\mathrm{n}} \leq M(x_1, \dots, x_{\mathrm{n}}; r(\omega)) = x_1 \vee \dots \vee x_{\mathrm{n}} = OR, \\ M(x_1, \dots, x_{\mathrm{n}};\ r(0)) = x_1 \wedge \dots \wedge x_{\mathrm{n}}, \\ M(x_1, \dots, x_{\mathrm{n}};\ r(1)) = x_1 \vee \dots \vee x_{\mathrm{n}}. \end{gathered} \qquad (4.5)$$

The function $M(x_1, \dots, x_{\mathrm{n}}; r(\omega))$ can be interpreted as a logic function: it has an adjustable degree of similarity to disjunction (or to conjunction) and represents a fundamental component for building GL.

In GL a name to this function M is: *graded* (or *generalized*) *conjunction/disjunction* (GCD) [7].

For GL, logic aggregators are functions that aggregate two or more degrees of truth and return a degree of truth in a way similar to observable patterns in human reasoning, in order to serve as mathematical models of human evaluation reasoning. Families of functions that are closely related to logic aggregators are means, and also, general aggregation functions, and triangular norms.

A function $f(x, y)$, to be considered a *mean*, and should have the following fundamental properties [7].

- Continuity: $\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} (f(x+\delta x, y+\delta y) = f(x, y)$,
- Internality: $\min(x, y) \le f(x, y) \le \min(x, y)$,
- Idempotency: $f(x, x) = x$,
- Symmetry: $f(x, y) = f(y, x)$,
- Homogeneity: $f(ax, ay) \le a\, f(x, y)$.

In a general case, we assume that symmetry is excluded because each argument may have a different degree of importance and commutativity is not desirable.

Definition 8. A *general aggregation function* (general aggregator) of n variables is a function A: $I^n \to I$ that is nondecreasing in each argument (monotonicity) and idempotent in extreme points 0 and 1(i.e. it must satisfy two boundary conditions) as follows:

$$\mathbf{x} = (x_1, \dots, x_n),\ \mathbf{y} = (y_1, \dots, y_n),\ x_i \in I,\ y_i \in I,\ i = 1, \dots, n$$

$$\mathbf{x} \le \mathbf{y} \Rightarrow A(\mathbf{x}) \le A(\mathbf{y}) \quad (\text{or } x_i \le y_i,\ i = 1, \dots, n \text{ implies } A(x_1, \dots, x_n) \le A(y_1, \dots, y_n) \qquad (4.7)$$

$$A(0, \dots, 0) = 0,\ A(1, \dots, 1) = 1$$

A general aggregator is defined less restrictively than a mean (internality, idempotency).

Mathematical literature, [10] uses the following classification of aggregators:

1) *Disjunctive* aggregators A: $1 \ge A(x_1, \dots, x_n) \ge \max(x_1, \dots, x_n) = x_1 \vee \dots \vee x_n$.
2) *Conjunctive* aggregators A: $0 \le A(x_1, \dots, x_n) \le \min(x_1, \dots, x_n) = x_1 \wedge \dots \wedge x_n$.
3) *Averaging* aggregators A: $\min(x_1, \dots, x_n) \le A(x_1, \dots, x_n) \le \max(x_1, \dots, x_n)$.
4) *Mixed* aggregators: aggregators that do not belong to groups 1), 2), 3).

But, in 1) – 4) the variables $x_1, \dots, x_n$ are not assumed to be degrees of truth of corresponding statements, and aggregators are not assumed to be functions of propositional calculus. Simultaneity (conjunctive aggregation) is recognized only in the lower region of the unit hypercube, and substitutability (disjunctive aggregation) is recognized only in the highest region of the unit hypercube. This is not consistent with the propositional logic interpretation of aggregation functions $[0, 1]^n \to [0, 1]$.

So, logic aggregators in GL are defined in a more restrictive way.

In GL classification of *logic aggregators* is based on the fact that basic logic aggregators are models of simultaneity or models of substitutability. The centroid of all logic aggregators is logic *neutrality*, modelled as the arithmetic mean. Therefore, assuming nonidentical arguments, the following basic classification of logic aggregators is used:

1) *Neutral logic aggregators*: $A(x_1, \dots, x_n) = MID(x_1, \dots, x_n) = (x_1 + \dots + x_n)/2$,
2) *Conjunctive aggregators*: $0 \le A(x_1, \dots, x_n) < MID(x_1, \dots, x_n)$,
3) *Disjunctive aggregators*: $MID(x_1, \dots, x_n) < A(x_1, \dots, x_n)$.

All nonidempotent conjunctive aggregators that satisfy $0 \le A(x_1, \ldots, x_n) < \min(x_1, \ldots, x_n)$ are denoted as hyperconjunctive. All nonidempotent disjunctive aggregators that satisfy $\max(x_1, \ldots, x_n) < A(x_1, \ldots, x_n) \le 1$ are denoted as hyperdisjunctive.

The areas of hyperconjunctive and hyperdisjunctive aggregators offer models of very high degrees of simultaneity and substitutability and overlap with the areas of triangular norms (t-norms, T) and triangular conorms (t-conorms, S), see, for example [4]. Among these aggregators, two of them, min/max (T_M, S_M), and product (T_P, S_P)) are sometimes used in logic aggregation for modelling very high levels of simultaneity and substitutability:

$$T_M(x, y) = \min(x, y), \qquad S_M(x, y) = \max(x, y),$$

$$T_P(x, y) = x \cdot y \quad , \qquad S_P(x, y) = x + y - x \cdot y.$$

Other t-norms and t-conorms have rather low applicability due to incompatibility with observable and proven properties of human reasoning, [7].

The concept of logic aggregator in GL is defined as consistent with observable and proven properties of human reasoning.

Definition 9. A *logic aggregator* $A(x_1, \ldots, x_n)$ is a continuous function of two or more variables $A: I^n \to I$ that satisfies the following additional conditions:

$\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $x_i \in I$, $y_i \in I$, $i = 1, \ldots, n$, $n > 1$;

$\mathbf{x} \le \mathbf{y} \Rightarrow A(\mathbf{x}) \le A(\mathbf{y})$, (nondecreasing monotonicity),

$A(0, \ldots, 0) = 0$, (boundary conditions for falsity),

$A(1, \ldots 1) = 1$, (boundary conditions for truth),

$A(x_1, \ldots, x_n) > 0$ if $x_i > 0$, $i = 1, \ldots, n$, (sensitivity to positive truth),

$A(x_1, \ldots, x_n) < 1$ if $x_i < 1$, $i = 1, \ldots, n$, (sensitivity to incomplete truth).

(The continuity of the logic aggregation function is requested, as well as two additional logic conditions.)

Having models of simultaneity and substitutability in mind and according to [6] the area of partial conjunction is located between the arithmetic mean and the pure conjunction $x_1 \wedge \ldots \wedge x_n = \min(x_1, \ldots, x_n)$, and the area of partial disjunction is located between the arithmetic mean and the pure disjunction $x_1 \vee \ldots \vee x_n = \max(x_1, \ldots, x_n)$. The intensity of partial conjunction (disjunction) is measured by andness (conjunction degree) α, (orness, disjunction degree, ω), [6], [7]:

$$\alpha = \frac{n - (n+1)\int_0^1 \cdots \int_0^1 A(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n}{n-1} \tag{4.8}$$

$$\omega = 1 - \alpha = \frac{(n+1)\int_0^1 \cdots \int_0^1 A(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n - 1}{n-1} \tag{4.9}$$

A high orness permits that a bad criterion be compensated by a good one. On the other hand, a high andness requires all criteria to be satisfied to a great degree. Andness and orness are related and add up to one. So, andness-directed transition from conjunction to disjunction (introduced in 1973 [6] to its current status [7]), is the history of an effort to interpret aggregation as a soft computing propositional calculus. Andness (orness) allows adjustable aggregators to be defined, with variable conjunction (disjunction) degree. When operators are between minimum and maximum, andness is for any number of inputs in the range [0, 1]. Operators that can return values smaller than the minimum (as t-norms) or larger than the maximum (as t-conorms) will provide andness outside [0, 1], reaching the minimum and the maximum of the interval with drastic disjunction and drastic conjunction [7].

The resulting analytic framework is a graded logic [7], based on analytic models of graded simultaneity (various forms of conjunction), graded substitutability (various forms of disjunction), and complementing (negation).

Definition 10. [7] *Graded logic* (GL) is an infinite-valued propositional calculus based on continuous, monotonic, noncommutative (weighted), and compensative models of graded simultaneity (conjunction) and graded substitutability (disjunction), and used primarily to create aggregation structures for computing the degree of truth of compound value statements. By combining the graded conjunction, disjunction, and standard negation, GL becomes a seamless generalization of the classic bivalent Boolean logic, extending it from $\{0, 1\}^n$ to $[0, 1]^n$.

The degree of truth of a value statement can be interpreted as the degree of suitability or the degree of preference. Suitability means the human percept of suitability. One of the main objectives is to compute the overall suitability of a complex object as a logic function of the suitability degrees of its components (attributes).

Basic graded logic functions can be *conjunctive*, *disjunctive*, or *neutral*. Conjunctive functions have andness (4.8) greater than orness (4.9), ($\alpha > \omega$). Similarly, disjunctive functions have orness greater than andness ($\alpha < \omega$), and neutral is only the arithmetic mean where $\alpha = \omega = 1/2$. Between the drastic conjunction and the drastic disjunction, we have andness-directed logic aggregators that are special cases of a fundamental logic function GCD [6]. GCD (symbol ◊) has the status of a logic aggregator, and it can be idempotent or nonidempotent, as well as *hard* (supporting annihilators) or *soft* (not supporting annihilators). The annihilator of hard conjunctive aggregators is 0, and the annihilator of hard disjunctive aggregators is 1.

The whole range of conjunctive aggregators is presented in Figure 12 [7].

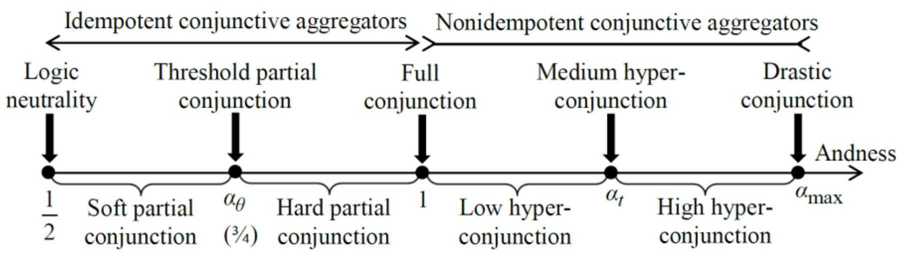

Figure 12. The range of conjunctive aggregators: border aggregators and aggregation segments.

A detailed classification of GCD aggregators, based on combinations of conjunctive/ disjunctive, idempotent/nonidempotent, and hard/soft aggregators is presented in Table 5 [7].

As both BL and GL share the same concept of duality (in De Morgan's sense) all disjunctive aggregators can be realized as De Morgan duals of conjunctive aggregators, so, it is sufficient to analyse only the conjunctive aggregators.

The interpolative method for implementation of GCD consists of implementing the border aggregators shown in Figure 12 and then using interpolative aggregators in the range of andness between them.

Taking into account that simplicity is the fundamental requirement for all aggregators, in GL the weighted power mean (WPM)

$$y = (w_1 x_1^r + \cdots + w_n x_n^r)^{1/r}, \quad -\infty \le r \le +\infty, \quad w_1 + \cdots + w_n = 1$$

is used as the main component, (the others exist, also) for building idempotent logic aggregator. The desired andness of the aggregator is easily adjusted by selecting the appropriate value

Table 5. Classification of andness-directed GL functions and aggregators.

		Logic function/aggregator	I	T	A	Global andness (α)	
GRADED LOGIC FUNCTIONS	CONJUNCTIVE	Drastic conjunction	N	H	0	$\alpha = \alpha_{max} = n / (n - 1)$	
		High hyperconjunction	N	H	0	$\alpha_t < \alpha < \alpha_{max}$	BASIC GL AGGREGATORS
		Medium hyperconjunction	N	H	0	$\alpha = \alpha_t = (n\ 2^n - n - 1) / (n - 1)\ 2^n$	
		Low hyperconjunction	N	H	0	$1 < \alpha < \alpha_t$	
		Full conjunction	Y	H	0	$\alpha = 1$	
		Hard partial conjunction	Y	H	0	$\alpha_\theta \leq \alpha < 1$; $1/2 < \alpha_\theta < 1$	
		Soft partial conjunction	Y	S	-	$1/2 < \alpha < \alpha_\theta$	
	Neutrality		Y	S	-	$\alpha = 1/2$	
	DISJUNCTIVE	Soft partial disjunction	Y	S	-	$1 - \alpha_\theta \leq \alpha < 1/2$	
		Hard partial disjunction	Y	H	1	$0 < \alpha \leq 1 - \alpha_\theta$	
		Full disjunction	Y	H	1	$\alpha = 0$	
		Low hyperdisjunction	N	H	1	$1 - \alpha_t < \alpha < 0$	
		Medium hyperdisjunction	N	H	1	$\alpha = 1 - \alpha_t$	
		High hyperdisjunction	N	H	1	$\alpha_{min} < \alpha < 1 - \alpha_t$	
		Drastic disjunction	N	H	1	$\alpha = \alpha_{min} = -1/(n - 1)$	

Columns: **I** = idempotent, **Y/N =** yes/no; **T =** type, **H/S** = hard/soft; **A** = annihilator

of exponent r, and the degrees of importance of attributes are selected using the normalized positive weights.

The most frequent human aggregation pattern of subjective categories reflects the condition for simultaneous satisfaction of two or more requirements. The degree of simultaneity (andness) can vary in a range from low to high, and partial conjunction (symbol Δ) is considered.

The other aggregation pattern (partial disjunction, symbol ∇) reflects the condition where the satisfaction of two or more requirements significantly satisfies an evaluation criterion, orness (degree of substitutability) can vary in the range of low to high.

The partial conjunction Δ has the andness α, dual partial disjunction ∇ has the orness ω. In GL standard negation is strictly used. In the case of partial conjunction and partial disjunction, aggregators Δ and ∇ are dual if $x\Delta y = \overline{\bar{x}\ \nabla \bar{y}}$ and $x\nabla y = \overline{\bar{x}\ \Delta \bar{y}}$, and the andness of Δ must be equal to the orness of ∇.

For weighted aggregators, if the partial conjunction Δ has the andness α, then the dual partial disjunction ∇ has the orness $\omega = \alpha$, so it follows:

$$\overline{W_1 x_1 \nabla \ldots \nabla W_n x_n} = W_1 \bar{x}_1 \Delta \ldots \Delta W_n \bar{x}_n \Rightarrow W_1 x_1 \nabla \ldots \nabla W_n x_n = 1 - W_1(1 - x_1)\ \Delta \ldots\ \Delta\ W_n(1 - x_n),$$
$$\overline{W_1 x_1 \Delta \ldots \Delta W_n x_n} = W_1 \bar{x}_1 \nabla \ldots \nabla W_n \bar{x}_n \Rightarrow W_1 x_1 \Delta \ldots \Delta W_n x_n = 1 - W_1(1 - x_1)\ \nabla \ldots\ \nabla\ W_n(1 - x_n).$$

WPM has asymmetric conjunctive and disjunctive properties, and no natural duality (exponential mean has natural duality). For WPM, De Morgan's laws are not naturally satisfied. Unless the asymmetric properties of WPM are not needed, it is necessary to redefine this aggregator using De Morgan's duals. If soft and hard aggregators are needed, then the dualized GDC aggregator can be defined as follows:

$$W_1 x_1 \Delta \ldots \Delta W_n x_n = (W_1 x_1^r + \cdots + W_n x_n^r)^{1/r},\ r = 1 - p,\ 0 \le p \le +\infty,$$
$$W_1 x_1 \nabla \ldots \nabla W_n x_n = 1 - W_1(1 - x_1)\ \Delta \ldots\ \Delta\ W_n(1 - x_n)$$
$$= 1 - (W_1(1 - x_1)^{2-p} + \cdots + W_n(1 - x_n)^{2-p})^{1/(2-p)},\ r = 1 + p,\ 0 \le p \le +\infty.$$

If $p = 0$, the resulting aggregator is the arithmetic mean. For $0 < p < 1$ this aggregator is a soft partial conjunction or the soft partial disjunction. For $1 \le p < +\infty$, the aggregator is a hard partial conjunction or the hard partial disjunction. The pure conjunction/disjunction are obtained for $p = +\infty$. The parameter p is used to adjust the desired values of andness and orness: the andness that corresponds to the exponent $r = 1 - p$ is the same as the orness that corresponds to the exponent $r = 1 + p$, $0 \le p \le +\infty$.

Compound functions can also have their De Morgan's duals [7].

Logical aggregators are practical models of observable human reasoning [7], and consequently, they belong to logic that is graded and defined as a strict generalization of classic Boolean logic. All aggregators that are related to human reasoning must be capable of modelling ten fundamental graded logic functions: (1) hyperconjunction, (2) conjunction, (3) hard partial conjunction, (4) soft partial conjunction (5) logic neutrality, (6) soft partial disjunction, (7) hard partial disjunction, (8) disjunction, (9) hyperdisjunction, and (10) negation. These functions are observable and provably present, in human reasoning; the graded logic conjecture [6] claims that they are both necessary and sufficient. Except for negation, all of them are special cases of the GCD aggregator (symbol ◊) which is a model of simultaneity and substitutability in GL. GCD and negation are observable in human intuitive reasoning, and necessary and sufficient to form a graded propositional calculus.

To be certified as a basic logic aggregator, an aggregator must satisfy a spectrum of conditions [7]. Ten core conditions include the following: (1) two or more input logic arguments that are degrees of truth and have clearly defined semantic identity, (2) the capability to cover the complete range of andness, making a continuous transition from drastic conjunction to drastic disjunction, (3) nondecreasing monotonicity in all arguments, and nonincreasing monotonicity in andness, (4) andness-directness for penalty or reward in the case of partial absorption aggregators, (5) importance weighting of inputs, (6) selectivity of conjunctive and disjunctive annihilators (0 and 1), (7) adjustability of threshold orness, (8) sensitivity to positive and incomplete truth, (9) absence of discontinuities and oscillatory properties, and (10) simplicity, readability, performance and the suitability for building compound aggregators. Except for GCD, huge set of aggregators were not introduced to create a complete system of logic functions with necessary human-centric properties.

Graded logic aggregators are indispensable components of most decision models. In the areas of logic aggregation and decision engineering, properties of human reasoning, generality, functionality and complexity are best satisfied by GCD aggregator [7]. In most practical problems GCD and negation create aggregation structures that efficiently implement expressions of graded propositional calculus.

From the theoretical basis of GL and aggregation, the Logic Scoring of Preference (LSP) Decision Engineering Framework (DEF) has been developed [7]. LSP DEF is developed for solving professional evaluation problems (i.e. problems that need a significant level of domain expertise), for example: the evaluation of computer systems, medical conditions, military equipment, complex software systems, urban plans, and ecological solutions. So, in many cases, domain experts are interested in evaluation methodology and evaluation problems solving using the LSP method.

Evaluation problems that are less dependent on professional expertise in a specific domain are usually personal decision problems, such as evaluating and selecting jobs, cars, homes, and schools for students. The LSP method has been successfully applied to those problems, also.

CONCLUSIONS

GL is a successor and seamless generalization of classical bivalent Boolean logic. All main properties of GL can be derived within the framework of classical logic, without explicitly using the concept of fuzzy set. On the other hand, the partial truth of a value statement can also be interpreted as a degree of membership of the evaluated object in a fuzzy set of maximum-value objects. So, the link between fuzzy logic and GL exists.

Relationships between the classical bivalent Boolean logic (BL), the graded logic (GL), and the fuzzy logic (FL) are subset-structured as follows: BL $\subset$ GL $\subset$ FL. BL is primarily a crisp bivalent propositional calculus. GL includes BL plus graded truth, graded idempotent conjunction/disjunction, weight-based semantics, and (less frequently) nonidempotent hyperconjunction/hyperdisjunction. GL also supports all nonidempotent basic logic functions (e.g. partial implication, partial equivalence, partial nand, partial nor, partial exclusive or, and others). All such functions are ”partial” in the sense that they use adjustable degrees of similarity or proximity (andness and orness) to their ”crisp” equivalents in traditional bivalent logic. FL includes GL, various forms of nonidempotent conjunction/disjunction, and other generalizations of multivalued logic. Fuzzy logic in a wide sense includes FL plus a wide variety of reasoning and computation based on the concept of fuzzy set.

GL is a descendant of both the BL and the FL. In the case of logic interpretation, all variables represent suitability, i.e. the degrees of truth of value statements that assert the highest values of evaluated objects or their components. In the case of fuzzy interpretation, the variables represent the degrees of membership in corresponding fuzzy sets of highest-value objects. In the case of bivalent logic, GL is a direct and natural seamless generalization of BL. (In points $\{0, 1\}^n$ of hypercube $[0, 1]^n$ we have BL = GL.) In the case of FL, GL is a special case because GL excludes various fuzzy concepts and techniques that are not related to logic. Since GL is primarily a propositional calculus, it is more convenient and more natural to interpret GL as a weighted compensative generalization of classical bivalent Boolean logic, than to interpret GL as a relatively narrow subarea in a heterogeneous set of models of reasoning and computation derived from the concept of fuzzy set.

GL is a soft logic based on GCD and means, and used primarily for evaluation. As opposed to that, in fuzzy logic, emphasis is not on evaluation problems. GL is a generalization of traditional Boolean logic based on concepts of graded conjunction and graded disjunction. FL is based on the graded concept of a fuzzy set, which is a generalization of the concept of a traditional crisp set. Figure 13 shows relations between various types of propositions and logics.

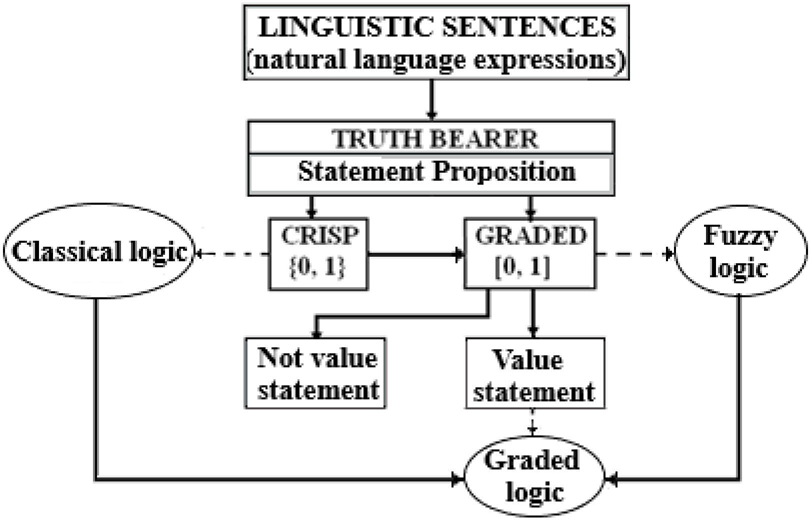

Figure 13. Relations between various types of propositions and logics.

The LSP method, developed from the theoretical basis of GL and aggregation, has been successfully applied to a plethora of problems.

Dujmović's results (beginning from [6], and later) are a strong contribution to the development and generalization of aggregation not only as part of fuzzy logic and soft computing. Those results, [6] to its current status [7], [8], improve Zadeh's approach in dealing with uncertain and vague information common in human reasoning, and also generate a special kind of logic, the graded logic. GL offers systematic, consistent controlled (by parameters) aggregators with gradual transitions from conjunctivity to disjunctivity, not only for use in FL, but GL is the whole one logic, accommodated to hierarchical structures characteristics for decision processes. GL brings to FL a consistently developed system of aggregators with a continuous transition between conjunction (simultaneity) and disjunction (substitutability) and the conditions of their partial validity required for all approaches that require aggregation. As GL and FL depend on degrees (GL on truth degree, FL on membership degree) so both are kind of graded logic in wide sense.

The topic discussed in the paper is of interest for many applications related to the development of intelligent systems: neural networks, vision systems, robotics, multicriteria decision-making systems in general, robotic networks (for example, platforms in smart cities [11], self-driving car networks, obviously [12]), and others, accommodated to automated hierarchical structures of decision processes closer to those processes humans apply.

And to explain the use of the word 'logics' in plural form in this article. Such somewhat unusual usage emphasizes multiple theories of logic, classical, fuzzy and graded logics, different logics to be used in different circumstances.

Development of GL shows that a significant insight into reality can still be achieved by carefully observing that reality (in time), without huge, expensive, experiments.

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